

Introduction to Evolution Equations in Geometry

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Preface

These are the very unpretentious lecture notes for the minicourse *Introduction to evolution equations in Geometry*, a part of the *Brazilian Colloquium of Mathematics*, to be held at IMPA, in July of 2009.

I have aimed at providing a first introduction to the main general ideas on the study of the Ricci flow , as well as guiding the reader through the steps of Kähler geometry for the understanding of the complex version of the Ricci flow . Most of the results concerning the (real) Ricci flow are due to Hamilton, and the Kähler Ricci flow results are mainly due to Cao, but while researching during the writing of my text, I found that the expositions in [CK04] and [Top06] are really clarifying.

There are extremely important and deep aspects to this theory, concerning Ricci flow with surgery and the work by Perelman, which are not discussed in this book.

I plan to keep improving these notes, and the updates will be available at www.math.duke.edu/~bsantoro. Meanwhile, I invite the reader to send suggestions, comments and possible corrections to

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A last remark is that the reader will notice that on this text, constants will be represented with the same symbol C , with rare exceptions.

Chapter 1

Introduction

A classical problem in geometry is to seek special metrics on a manifold. From our differential geometry classes, we recall that the topology of the manifold plays a central role in determining the presence of metrics with special curvature. For the simple case of surfaces, the **Gauss-Bonnet Theorem** relates the total curvature of a surface M with its Euler characteristic:

$$\int_M K dS = 2\pi\chi(M).$$

This expression tells us that every time we try to "flatten" a region of our surface, we must pay the price of modifying the curvature somewhere else as well. Another direct consequence is that a sphere will never admit a flat metric.

In dimension 2, the **Uniformization Theorem** states that every surface admits a metric of constant curvature $-1, 0$ or 1 , according to its genus.

It is hence a natural question to ask if such a classification extends to higher dimensions.

Already in dimension 3, a naive attempt of an immediate generalization of the Uniformization Theorem fails, as $S^2 \times S^1$ cannot admit a metric of constant curvature. So, a classification would only be possible if we broaden the conditions, as well as narrow the types of manifolds in question.

As we shall see in Chapter 2, Thurston's Decomposition Theorem tells us that, for any 3-manifold M , there is a way to cut it along special tori and spheres (and gluing 3-balls in the remaining boundary pieces) in such a way that the remaining pieces admit "canonical geometries", in a way to be made precise in later chapters.

And here the Ricci flow enters the stage. Hamilton suggested that such a decomposition in pieces with nice geometric structures could be obtained by flowing any initial metric on a manifold through a smartly chosen equation. Understanding the limiting metrics and the structure of eventual singularities that may occur would provide a complete understanding of the problem.

Now, what is a good choice for a flow? The strategy to produce a nice evolution equation is to minimize the right choice of energy functional. Keeping in mind that any expression in the first derivatives of g would vanish in normal coordinates, a first choice for such an energy functional would be

$$E = \int_M R d\mu,$$

where R denotes the scalar curvature of a metric g .

The associated flow equation for minimizing E is

$$\frac{\partial}{\partial t} g(t) = \frac{2}{n} R g(t) - 2 \text{Ric}(t),$$

where Ric is the Ricci curvature of the metric $g(t)$.

Unfortunately, such an equation would imply that the evolution equation for the scalar curvature is given by

$$\frac{\partial}{\partial t} R = - \left(\frac{n}{2} - 1 \right) \Delta R + |\text{Ric}|^2 - \frac{1}{2} R^2,$$

which is a backwards heat equation on R , which may have no solutions even for short time. Therefore, we need to be smarter about the choice of evolution equation.

Consider

$$\begin{cases} \frac{\partial}{\partial t} g(t) = \frac{2}{n} r g(t) - 2 \text{Ric}(t) \\ g(0) = g_0, \end{cases} \quad (1.1)$$

where r denotes the average of the scalar curvature on the manifold M .

In fact, the first factor in the right-hand side of (1.1) is just a normalization that will guarantee that the volume of the manifold remains constant along the flow. We claim that solutions to (1.1) correspond (via rescalings) to solutions to the unnormalized Ricci flow equation

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2\text{Ric}(t) \\ g(0) = g_0. \end{cases} \quad (1.2)$$

To see this, let g be a solution to (1.2), and choose a function ψ such that the metric $\tilde{g} = \psi g$ has volume $\int d\tilde{\mu} = 1$. With this normalization, you can check that

$$\tilde{\text{Ric}} = \text{Ric}, \quad \tilde{R} = \psi^{-1}R, \quad \tilde{r} = \psi^{-1}r,$$

where the geometric objects denoted with a \sim on top are the ones related to \tilde{g} .

Let $\tilde{t} = \int \psi(t)dt$ be a rescaling of time. Then,

$$\frac{\partial}{\partial \tilde{t}}\tilde{g} = \frac{\partial}{\partial t}g + \left(\frac{d}{dt}\log\psi\right)g.$$

Also, according to our definition,

$$\int d\mu = \psi^{-n/2} \quad \text{and} \quad \frac{\partial}{\partial t}\log\mu = \frac{1}{2}g^{ij}\frac{\partial}{\partial t}g_{ij} = -R$$

and since

$$\frac{d}{dt}\log\left(\int d\mu\right) = -r,$$

this implies that

$$\frac{\partial}{\partial \tilde{t}}\tilde{g} = \frac{2}{n}\tilde{r}\tilde{g} - 2\tilde{\text{Ric}},$$

i.e., \tilde{g} is a solution to the normalized Ricci flow (1.1).

In dimension 2, the Ricci flow has a solution that exists for all times, and it converges to the special metric of constant curvature of the surface, providing a different proof for the Uniformization Theorem.

More generally, the Ricci flow may develop singularities in finite time. Nevertheless, the behavior serves for understanding better the underlying topology.

The current strategy is: we stop the flow once the singularity occurs, perform a very careful surgery on the manifold and we restart the flow on the "desingularized" manifold. Thanks to the groundbreaking work of Perelman [Pera], [Perb], it is known that this surgery procedure will only be performed a finite number of times, and will provide the Thurston's decomposition. Due to the introductory flavor of this set of notes, we will not devote time to explain the beautiful theory of singularities on Ricci flow. We refer the reader to [CCG⁺07] for a very good exposition of this material.

In order to develop some very basic intuition to the Ricci flow, let us study the simple example of a round sphere, with a metric g_0 such that

$$\text{Ric}(g_0) = \lambda g_0$$

for some constant $\lambda > 0$. We invite the reader to check that the solution of the Ricci flow equation is

$$g(t) = (1 - 2\lambda t)g_0.$$

So, we see that the unnormalized Ricci flow shrinks a round sphere to a point in time $T = (2\lambda)^{-1}$.

Note also that if g_0 were a hyperbolic metric (of constant sectional curvature -1), then

$$\text{Ric}(g_0) = -(n-1)g_0,$$

and so

$$g(t) = (1 + 2(n-1)t)g_0,$$

which shows that the manifold expands homothetically for all times.

For a beautiful exposition on the intuition for the Ricci flow, we refer the reader to [Top06].

Chapter 2

The Geometrization Conjecture

Our goal for this chapter is to describe the concept of a model geometry, understand Thurston's classification of homogeneous geometries, and state the Geometrization Conjecture in detail.

2.1 Introduction

In order to provide a better picture, we shall describe the concept of a model geometry for surfaces.

Let X be either S^2 (the sphere), \mathbb{E}^2 (Euclidean plane) or \mathbb{H}^2 (hyperbolic plane).

Definition 2.1.1. *Let F be a closed surface. If F can be described as the quotient X/Γ , where Γ is a subgroup of isomorphisms of X , such that the map $X \rightarrow F$ is a covering space, then we say that F has a **geometric structure modeled on X** .*

The **Uniformization Theorem** tells us that every closed surface admits a geometric structure, and due to Gauss-Bonnet Theorem, this structure must be unique, and is determined by the Euler characteristic of the surface.

A natural question that arises is whether this classification can be extended to higher dimensions. The simple example of $S^2 \times S^1$ tells us that a naive extension of such classification fails, for it cannot be covered by any of \mathbb{E}^3 , S^3 or \mathbb{H}^3 , as the stabilizer of a point is not in $O(3)$.

First, we need to explain what do we mean by a geometric structure.

Definition 2.1.2. *A metric on a manifold M is **locally homogeneous** if for any $x, y \in M$, there exist neighborhoods U of x and V of y , and an isometry $U \rightarrow V$ mapping x to y .*

Intuitively, a locally homogeneous manifold looks the same in a neighborhood of any point.

Definition 2.1.3. *We say that M **admits a geometric structure** if M admits a complete, locally homogeneous metric.*

It is a theorem by Singer [Sin60] that if a manifold happens to be simply connected, then being locally homogeneous is equivalent to being homogeneous. Therefore, up to passing to the universal cover, we may restrict ourselves to the study of homogeneous geometries.

However, many manifolds do not admit a geometric structure. For example, any non-trivial connected sum does not admit a geometric structure, with the exception of $\mathbb{P}^3 \# \mathbb{P}^3$. So we should be a little more subtle in our attempt to classify 3-manifolds.

The famous theorem by Thurston roughly says that any compact, orientable 3-manifold M can be cut along disjoint embedded 2-spheres and tori in such a way that after gluing 3-balls to all boundaries, each piece admits a geometric structure. We will describe this statement in further detail on the next section.

For a detailed description of the eight geometries, we refer the reader to [Sco83].

2.2 Thurston's Classification of 3-dimensional geometries

In this section, we would like to present an outline of the proof that there are only 8 distinct geometric structures, and that once a man-

ifold admits one of them, it is unique.

Let M be a closed 3-manifold, and let X be its universal cover. As we have seen, M admits a geometric structure if X has a complete homogeneous metric. For this case, the isometry group of X acts transitively, with compact point stabilizers.

Definition 2.2.1. *Throughout this chapter, a **geometry** shall mean a pair (X, G) , where X is a simply connected manifold, and G acts on X with compact point stabilizers.*

We will only consider maximal geometries (when G is maximal), and we need to impose the condition that there must exist a subgroup H of G which acts on X as a covering group and has compact quotient.

Theorem 2.2.1. *(Thurston) Any maximal, simply connected 3-dimensional geometry which admits a compact quotient is equivalent to a pair (X, G) , $G = \text{Isom}(X)$, where X is one of the following:*

$$\mathbb{E}^3, S^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, SL(2, \mathbb{R}), Nil, Sol.$$

The remainder of this section will be devoted to the proof of this result.

Given (X, G) , endow X with a G -invariant metric g , via the classical construction: on the tangent space of a point $T_x X$, build a G -invariant inner product by taking the average over the stabilizer, and use the homogeneity of the space to extend the metric to the whole manifold. The (simple) details on this construction are left to the reader.

Choose a point $x \in X$, and consider $I(G_x)$, the identity component of the stabilizer of G . Clearly, G_x acts on $T_x X$, preserving the inner product on $T_x X$ given by the G -invariant metric g . Therefore, G_x must be a compact subgroup of $O(3)$.

Since $I(G_x)$ is connected, there are not many choices for it: it can either be the trivial group, $SO(2)$ or $SO(3)$. This subdivision plays a crucial role in the classification

- **Case 1:** If $I(G_x) = SO(3)$.

The manifold looks identical in every direction, so it can be only \mathbb{E}^3 , \mathbb{H}^3 or S^3 . These cases can be differentiated according to the sign of the sectional curvature, constant in every direction.

- **Case 2:** If $I(G_x) = SO(2)$.

Let $L_x \subset T_x X$ be the 1-dimensional subspace fixed by $SO(2)$, and let P_x be the orthogonal complement (with respect to the G -invariant metric g) of L_x in $T_x X$.

Since $I(G_x)$ is a normal subgroup, both L_x and P_x are invariant under the action of G_x . Furthermore, since X is simply-connected, we can choose a coherent orientation to let L_x define a unit vector field v_x on X .

Note that the vector field does not need to be invariant under the whole group G , but it will be invariant under some subgroup G_1 of G of order at most 2. Also the plane field defined by P_x must be also invariant under G_1 . Therefore, both v_x and P_x descend to any manifold covered by X and with covering group contained in G_1 .

Let $\phi_t(\cdot)$ be the flow generated by v_x , and note that it also descends to the covered manifold. The plane field P_x inherits an inner product from $T_x X$, which is preserved by ϕ_t .

To see this, take any vector $u \in P_x$, and set $\|d\phi_t(u)\| = f(t)\|u\|$. Since the vector field is preserved by the flow, the volume form will get multiplied by a constant factor $f(t)$. Since the manifold is compact, $f(t) = 1$.

Now, if the plane field is integrable, X must be either $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{E}^2 \times \mathbb{R}$. The latter will be disregarded, since we are only looking for maximal geometries.

If P_x is not integrable, then X must be isometric to either $(\tilde{SL}(2, \mathbb{R}))$, Nil or S^3 , which will also be discarded for maximality.

- **Case 3:** If $I(G_x) = G_e$ is trivial.

This means that G_e acts transitively and freely on X , so we can identify X with the group G_e itself. Since the geometry (X, G) was assumed to admit a compact quotient, it follows that G_e has a subgroup H such that the quotient G_e/H is compact.

Definition 2.2.2. A Lie Group G is **unimodular** if its left-invariant Haar measure is also right-invariant.

If G has a discrete subgroup H such that $H \backslash G$ has a finite measure inherited from the left invariant Haar measure on G , then G must be unimodular. Therefore, our G_e must be unimodular.

Milnor [Mil76] classified all simply connected, unimodular Lie Groups: S^3 , \mathbb{E}^3 , $SL(\tilde{2}, \mathbb{R})$, $\text{Isom}(\tilde{\mathbb{E}^2})$, Nil and Sol .

Since we are requesting that (X, G) is a maximal geometry, the only possibility left is that $X = Sol$. We must check that for any left-invariant metric on Sol , $I(G_x)$ is trivial. The argument we follow is provided by Milnor, in [Mil76]. He shows that a left-invariant metric on Sol determines in a canonical way an orthonormal basis of the tangent space at the identity. Such basis consists of eigenvectors of a naturally defined self-adjoint map on the tangent space at the identity, with associated eigenvalues $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_3 < 0$.

Hence, any isometry of Sol which fixes the identity must satisfy very specific conditions on the eigenvectors. One can then deduce that the stabilizer of a point has order at most 8, and if it has order exactly 8, it must be isomorphic to $D(4)$, which can be realized via the metric we described on last section.

This completes the proof of Theorem 2.2.1.

The proof of the following theorem, together with a very nice survey on 3-dimensional geometries, can be found in [Sco83].

Theorem 2.2.2. *If M is a closed 3-manifold which admits a geometric structure modelled in one of the eight possible model geometries, then it must be unique.*

Chapter 3

Ricci Flow on Homogeneous Geometries

This chapter will be devoted to the study of the behavior of Ricci flow on homogeneous geometries. This will help the reader to build up some intuition for the Ricci Flow, as in this case, the partial differential equation will be simplified into a system of coupled ordinary differential equations due to homogeneity of the space. We will follow closely the approach chosen by [CK04], as it exemplifies very well the main character of this evolution equation.

We will restrict our attention to the case of the five model geometries that can be realized as a pair (G, G) , where G is a simply-connected, unimodular Lie Group.

3.1 Ricci Flow as a system of ODE's

Let G be any Lie Group, and let \mathcal{G} be its Lie Algebra. The set of left-invariant metrics on G can be naturally identified with the set S_n of symmetric $n \times n$ -matrices. For each metric, the Ricci flow defines a path $g(t) \in S_n$, so we immediately see the reduction of the flow to a

system of $n(n+1)/2$ equations. However, by using a smart choice of moving frames (the **Milnor Frame**), we will be able to diagonalize the system, as follows.

First, we will equip \mathcal{G} , the Lie Algebra of a 3-dimensional Lie Group G , with a left-invariant moving frame $\{F_i\}$, $i = 1, 2, 3$. We define structure equations by

$$[F_i, F_j] = c_{ij}^k F_k, \quad (3.1)$$

and the **adjoint representation** of \mathcal{G} is the map $ad : \mathcal{G} \rightarrow gl(\mathcal{G})$ defined by

$$ad(V)(W) = [V, W]. \quad (3.2)$$

Once the orthonormal frame $\{F_i\}$ is fixed, we define an endomorphism $\mathcal{G} \rightarrow \mathcal{G}$ by

$$F_1 \mapsto [F_2, F_3], \quad F_2 \mapsto [F_3, F_1], \quad F_3 \mapsto [F_1, F_2]. \quad (3.3)$$

For our fixed basis, the matrix C representing this endomorphism has the structure constants as its entries.

Our goal is to argue that there exists a choice of frame that makes the matrix C be diagonal. Observe that if G is unimodular, then $tr(ad(V)) = 0$ for every $V \in \mathcal{G}$. It is a very simple exercise to check that this condition implies that the matrix C is necessarily self-adjoint, and hence there exists an orthogonal change of basis such that the matrix \hat{C} representing the endomorphism in the new basis is orthogonal.

By reordering, and dropping the assumption that the new frame must be **orthonormal** (we will only need to keep it orthogonal), we may assume that the matrix for this endomorphism is given by

$$C = \begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\nu \end{pmatrix},$$

where $\lambda \leq \mu \leq \nu$, and $\lambda, \mu, \nu \in \{-1, 0, 1\}$. This choice of frame is called a **Milnor Frame**.

Now, let $\{F_i\}$ be a Milnor frame for some left-invariant metric g . Then, there exist $A, B, C > 0$ such that g can be written as

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$$

using the dual frame $\{\omega^i\}$.

In order to evolve the metric g by the Ricci flow, we will compute explicitly its curvature. Recall that the **Levi-Civita connection** of the metric g is given by

$$\nabla_X Y = \frac{1}{2} \{[X, Y] - (adX)^*Y - (adY)^*X\}, \quad (3.4)$$

and once we know this expression, we can write the curvature tensor R of the metric g as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (3.5)$$

Now, note that if $\{F_i\}$ is a Milnor frame, then it is a simple computation to check that the map ad^* is determined by

$$ad^* = ((adF_i)^* F_j)_i^j = \begin{pmatrix} 0 & 2\lambda \frac{A}{C} F_3 & -2\lambda \frac{A}{B} F_2 \\ -2\mu \frac{B}{C} F_3 & 0 & 2\mu \frac{B}{A} F_1 \\ 2\nu \frac{C}{B} F_2 & -2\nu \frac{C}{A} F_2 & 0 \end{pmatrix}$$

Lemma 3.1.1. *For our choice of a Milnor frame, we have that*

$$\langle R(F_k, F_i)F_j, F_k \rangle = 0.$$

The proof of this lemma is obtained immediately by our explicit expressions for the Levi-Civita connection and curvature.

This result tells us that a choice of Milnor frame allows us to identify globally both g and $Ric(g)$ with diagonal matrices. So the Ricci flow is reduced to a system of only 3 equations, instead of 6, as expected.

Having completed the necessary background, we will present the behavior of the flow of homogeneous metrics with the 3 possible isotropy groups.

3.2 Ricci Flow of geometry with isotropy $SO(3)$

This group includes $\mathbb{R}^3, \mathbb{H}^3$ and S^3 . We will restrict our attention to S^3 , to be identified with the Lie Group $SU(2)$ of complex 2×2

matrices with determinant equal to 1. The signature for a Milnor frame on $SU(2)$ is given by $\lambda = \mu = \nu = -1$.

We will study the behavior of the Ricci Flow on $SU(2)$ with respect to a 1-parameter family of initial data exhibiting collapse (in the Gromov-Hausdorff sense) to a lower dimensional manifold.

Consider the **Hopf Fibration** $S^1 \mapsto S^3 \rightarrow S^2$ induced by the projection $\pi : SU(2) \rightarrow \mathbb{CP}^1$ given by $\pi(w, z) = [w, z]$. In [Ber60], Berger exhibit a collapse of S^3 to a round S^2 while keeping the curvature bounded. Following [CK04], we study here a family of left-invariant initial metrics $\{g_\varepsilon\}$, $\varepsilon \in (0, 1]$, on $SU(2)$. With respect to a fixed Milnor Frame, we will write these metrics as

$$g_\varepsilon = \varepsilon A \omega^1 \otimes \omega^1 + B \omega^2 \otimes \omega^2 + C \omega^3 \otimes \omega^3 \quad (3.6)$$

A simple exercise is left to the reader: using the expression for the Riemann tensor in terms of the structural constants, together with the definition of **sectional curvature** of the plane spanned by F_i and F_j ,

$$K(F_i, F_j) = \langle R(F_i, F_j)F_j, F_i \rangle,$$

you may check that the Ricci tensor of the metric g_ε is determined by

$$\text{Ric}(F_1, F_1) = \frac{2}{BC} [(\varepsilon A)^2 - (B - C)^2] \quad (3.7)$$

$$\text{Ric}(F_2, F_2) = \frac{2}{\varepsilon AC} [B^2 - (\varepsilon A - C)^2] \quad (3.8)$$

$$\text{Ric}(F_3, F_3) = \frac{2}{\varepsilon AB} [C^2 - (\varepsilon A - B)^2] \quad (3.9)$$

The following proposition tells us that independently of the choice of a initial homogeneous metric on $SU(2)$, the Ricci Flow will shrink to a round point in finite time.

Proposition 3.2.1. *For any $\varepsilon \in (0, 1]$ and any choice of initial data $A_0, B_0, C_0 > 0$, the solution g_ε of 1.2 exists for a maximal finite time $T_\varepsilon < \infty$, becoming asymptotically round.*

Proof. The proof we outline here is provided in further detail in [CK04]. It has two main steps: we first notice that (3.7) determine

the right-hand side of the system for the evolution of A , B and C . Due to the symmetry of the system, we can assume that $\varepsilon A_0 \leq C_0 \leq B_0$.

It is simple to check that

$$\frac{d}{dt} \log(B - \varepsilon A) = 4 \frac{C^2 - (B + \varepsilon A)^2}{\varepsilon ABC},$$

which shows that the relation $\varepsilon A \leq C \leq B$ is preserved under the flow. Furthermore, note that

$$\frac{\partial}{\partial t} B \leq -8 + \frac{4\varepsilon A}{C} \leq -4,$$

so the solution can only exist on a finite time T_ε .

The second step is to show that the metric becomes round as $t \rightarrow \infty$. One can check that

$$\frac{\partial}{\partial t} \frac{B - \varepsilon A}{\varepsilon A} \leq 0,$$

so this quantity is bounded above by its initial value. Therefore, for $\delta = \frac{B_0 - \varepsilon A_0}{\varepsilon A_0}$, we have

$$0 < B - \varepsilon A \leq \delta \varepsilon A,$$

completing the proof of the proposition. \square

In contrast with the other model geometries, the Ricci flow of a homogeneous metric will avoid collapse. We will refer to [CK04] for the proof that in fact, the maximal times T_ε can be chosen uniformly for all $\varepsilon \in (0, 1]$.

For the result about collapsing, we define the quantities $E = B + C$, and $F = (B - C)/\varepsilon$.

Proposition 3.2.2. *If $g_\varepsilon(t)$, a family of metrics parametrized by $\varepsilon \in (0, 1]$, satisfies*

$$\lim_{\varepsilon \rightarrow 0} F(0, \varepsilon) > 0,$$

then for all $t \in (0, T_{max})$,

$$\lim_{\varepsilon \rightarrow 0} F(t, \varepsilon) = 0.$$

This proposition shows that there is a jump discontinuity at $t = 0$.

Proof. We just present here an outline. The evolution equation for F is given by

$$\frac{\partial}{\partial t} \log F = \frac{4}{ABC} \left(\varepsilon A^2 - \frac{E^2}{\varepsilon} \right).$$

So, the proof follows from noticing that

$$\frac{4}{ABC} \left(\varepsilon A^2 - \frac{E^2}{\varepsilon} \right) \rightarrow \infty \quad \text{as} \quad \varepsilon \rightarrow 0.$$

□

3.3 Ricci Flow of geometry with isotropy $SO(2)$

The geometries contained in this group are $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil and $SL(2, \mathbb{R})$. The former has a distinct behavior amongst the others: under the Ricci flow, the \mathbb{R} -factor remains fixed, while the S^2 -factor shrinks. Therefore, the solution becomes singular in finite time and converge to a 1-dimensional manifold.

For the remaining geometries, some of its directions get expanded by the flow, while some others will converge to a finite value. For this case, we will study in detail the Lie Group Nil , as all the geometric objects are simple to compute, and it captures the main behavior of the Ricci flow on $SO(2)$ isotropy.

Let G denote the 3-dimensional group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

The signature of a Milnor frame for G is $\lambda = -1$, $\mu = \nu = 0$. As before, we can write any left-invariant metric g on G as

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3$$

with respect to a Milnor frame of 1-forms $\{\omega_i\}$.

Also, the reader can check that the Ricci tensor of the metric g will be given by

$$\text{Ric} = \frac{2A^2}{BC}\omega^1 \otimes \omega^1 - \frac{2A}{C}\omega^2 \otimes \omega^2 - \frac{2A}{B}\omega^3 \otimes \omega^3.$$

Hence, the evolution equations for A, B and C are given by

$$\frac{\partial}{\partial t}A = \frac{-4A^2}{BC}; \quad \frac{\partial}{\partial t}B = \frac{4A}{C}; \quad \frac{\partial}{\partial t}C = \frac{4A}{B}.$$

We are lucky enough to have explicit solutions to this system, given by

$$A(t) = A_0^{2/3} B_0^{1/3} C_0^{1/3} (12t + B_0 C_0 / A_0)^{-1/3} \quad (3.10)$$

$$B(t) = A_0^{1/3} B_0^{2/3} C_0^{-1/3} (12t + B_0 C_0 / A_0)^{1/3} \quad (3.11)$$

$$C(t) = A_0^{1/3} B_0^{-1/3} C_0^{2/3} (12t + B_0 C_0 / A_0)^{1/3} \quad (3.12)$$

So, we can see explicitly that, for the *Nil* geometry, the Ricci flow will expand two of the directions determined by the Lie algebra, while the other will converge to a fixed value. As a corollary, we see that any compact manifold with *Nil* geometry will converge, via the normalized Ricci flow, to \mathbb{R}^2 endowed with the flat metric.

3.4 Ricci Flow of geometry with trivial isotropy

The only geometry in this category is *Sol*, and the signature of a Milnor frame on this group is $\lambda = -1$, $\mu = 0$ and $\nu = 1$. Using this quantities, we obtain the following evolution equations for the coefficients of a left-invariant metric g (with respect to a Milnor frame):

$$\frac{\partial}{\partial t}A = 4 \frac{C^2 - A^2}{BC} \quad (3.13)$$

$$\frac{\partial}{\partial t}B = 4 \frac{C^2 + A^2}{AC} \quad (3.14)$$

$$\frac{\partial}{\partial t}C = 4 \frac{-C^2 + A^2}{AB}. \quad (3.15)$$

Note that $C \frac{\partial}{\partial t} A + A \frac{\partial}{\partial t} C = 0$, so the quantity AC is preserved. So, we can define $G = A/C$, and rewrite the system above as

$$\frac{\partial}{\partial t} B = 8 + 4 \frac{1 + G^2}{G} \quad (3.16)$$

$$\frac{\partial}{\partial t} G = 8 \frac{1 - G^2}{B}. \quad (3.17)$$

We have that $\frac{\partial}{\partial t} B \geq 16$, so $B(t) \rightarrow \infty$. It is also true that $G(t) \rightarrow 1$ as $t \rightarrow \infty$, and in fact, both A and C will converge to the same value.

Therefore, our conclusion is that the solution $g(t)$ to the Ricci flow exists for all times. Furthermore, if we consider the normalized Ricci flow on a compact 3 manifold modeled on Sol , we will observe convergence to \mathbb{R} in the Gromov-Hausdorff sense.

Chapter 4

Ricci Flow on Surfaces

In this chapter, we will explain the behavior of the Ricci flow on compact surfaces. In dimension 2, the Ricci flow equation is simply given by

$$\frac{\partial}{\partial t}g(t) = -Rg. \quad (4.1)$$

In order to keep the total area A of the surface constant, we will introduce an extra constant in the equation, obtaining the **normalized Ricci flow for surfaces**:

$$\frac{\partial}{\partial t}g(t) = (r - R)g, \quad (4.2)$$

where $r = \frac{4\pi\chi(M)}{A}$ ¹ is the average of the scalar curvature on the surface.

Note that, at each point, the rate of change is a multiple of the metric, so we will be flowing the metric inside its conformal class. In fact, this equation makes sense even in higher dimensions, and is known as the **Yamabe flow**. The Yamabe flow is the gradient flow related to the so-called **Yamabe problem**: fixed a conformal class of a metric on a compact manifold, as well as its volume, is there a metric of constant scalar curvature in the fixed conformal class?

In a sense, the 2-dimensional Ricci flow resembles much more of the character of the Yamabe flow. Nevertheless, we hope that the

¹This is a direct application of Gauss-Bonnet theorem.

study of the Ricci flow on surfaces will be a good introduction to the general and far more complicated case of 3 dimensions.

The goal of this chapter is to prove the following results.

Theorem 4.0.1. (Hamilton) *For any initial data, the solution to the normalized Ricci flow equation (4.2) exists for all times.*

Theorem 4.0.2. (Hamilton) *If $r \leq 0$, the metric converges to a constant curvature metric.*

Theorem 4.0.3. (Hamilton) *If $R > 0$, the metric converges to a constant curvature metric.*

An important observation is that from the work of Hamilton, one cannot deduce directly the Uniformization Theorem for surfaces. Even though it treats successfully the cases where $\chi(M) < 0$ and $\chi(M) = 0$, it uses the fact that there exists a metric of constant positive curvature on S^2 .

Recently, however, Chen, Lu and Tian [CLT06] provided a pure Ricci flow proof of the Uniformization conjecture. We will come back to this at the end of the chapter.

4.1 Some estimates

In order to study the Ricci flow on surfaces, we start by developing the evolution equation for the scalar curvature $R(t)$ of the metric $g(t)$. As previously noted, the metric $g(t)$ will be always a conformal deformation of the initial metric g_0 . So, it is useful to develop the relation between the scalar curvatures of two conformal metrics.

Lemma 4.1.1. *If g and $h = e^{2u}g$ are two conformal metrics, then*

$$R_h = e^{-2u}(-2\Delta_g u + R_g) \quad (4.3)$$

Before we present the proof of the lemma above, let us recall the technique of **moving frames**. Let $\{\omega_1, \omega_2\}$ be a coframe field, chosen to be orthonormal with respect to the metric g . So, we can write the metric g as

$$g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2.$$

Let $\{f_1, f_2\}$ be the dual frame (*i.e.*, $f_i \omega^j = \delta_i^j$).

We define the **connection 1-forms** ω_i^j by the following relation: let X be a vector field. Then,

$$\nabla_X f_i = \omega_i^j f_j.$$

Using this, we can write the Cartan structure equations for a surface as

$$d\omega^1 = \omega^2 \wedge \omega_2^1; \quad (4.4)$$

$$d\omega^2 = \omega^1 \wedge \omega_1^2; \quad (4.5)$$

$$\Omega_2^1 = d\omega_2^1. \quad (4.6)$$

Proof. As before, let $\{f_1, f_2\}$ be a orthonormal frame for g , dual to $\{\omega_1, \omega_2\}$. Then

$$e_1 = e^{-u} f_1 \quad e_2 = e^{-u} f_2$$

is an orthonormal moving frame for h . Call

$$\eta_1 = e^u \omega_1 \quad \eta_2 = e^u \omega_2$$

the coframe for h . The proof follows from:

- Writing $d\eta^1$ and $d\eta^2$ in terms of u , ω^1 and ω^2 ;
- Computing

$$\eta_2^1 = d\eta^1(e_2, e_1)\eta^1 + d\eta^2(e_2, e_1)\eta^2,$$

and applying to the second structure equation of Cartan (4.4);

- Noticing that, in an orthonormal frame, $\Delta_g u = \nabla_{f_1, f_1}^2 u + \nabla_{f_2, f_2}^2 u$, which yields

$$(\Omega_h)_2^1 = (\Omega_g)_2^1 - \Delta_g u \omega^1 \wedge \omega^1;$$

- Finally, recalling that

$$R_h = 2(\Omega_h)_2^1 = e^{-2u}(R_g - 2\Delta_g u).$$

□

Now, we study the evolution of the curvature $R(t)$ under the Ricci flow.

Lemma 4.1.2. *If $g(t)$ is a 1-parameter family of metrics on a surface M^2 such that*

$$\frac{\partial}{\partial t}g = fg$$

for some function f , then

$$\frac{\partial}{\partial t}R = -\Delta f - Rf.$$

In particular, if $f = r - R$ (the normalized Ricci flow equation), then

$$\frac{\partial}{\partial t}R = \Delta R + R(R - r). \quad (4.7)$$

Proof. We have already seen the relation between the scalar curvatures of two conformal metrics g and $h = e^{-2u}g$ is given by (4.3). So, in that case, if $\frac{\partial}{\partial t}g = fg$, then $\frac{\partial}{\partial t}u = f$. Differentiating (4.3) with respect to time, we obtain

$$\frac{\partial}{\partial t}R_h = -\left(\frac{\partial}{\partial t}u\right)e^{-u}(-\Delta_g u + R_g) - e^{-u}\Delta_g\left(\frac{\partial}{\partial t}u\right) = -\Delta_h f - R_h f.$$

□

4.1.1 Scalar Maximum Principle

At this point, we will start getting into the spirit of geometric analysis. We would like to establish some bounds for the scalar curvature of a metric running under the Ricci flow. Our strategy is to compare a solution of the PDE

$$\frac{\partial}{\partial t}R = \Delta R + R(R - r)$$

with the solution of the associated ODE to this problem, obtained by ignoring the Laplacian term on the equation above. In order to

do so, we need to use a version of the scalar maximum principle for a heat equation with non-linear reaction term.

For completeness, we state here the version of the Maximum Principle that we will use. Consider the following (more general) equation for the evolution of a scalar quantity v :

$$\frac{\partial}{\partial t}v = \Delta_{g(t)}v + F(v), \quad (4.8)$$

where $g(t)$ is a 1-parameter family of metrics, and $F : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function.

The function u is said to be a **supersolution** to (4.8) if

$$\frac{\partial}{\partial t}u \geq \Delta_{g(t)}u + F(u)$$

and a **subsolution** if

$$\frac{\partial}{\partial t}u \leq \Delta_{g(t)}u + F(u).$$

Theorem 4.1.1. (Scalar Maximum Principle) *Let u be a C^2 supersolution to (4.8) on a closed manifold M . Suppose that there exists $C_1 > 0$ such that $u(x, 0) \geq C_1$ for all x in M , and let ϕ_1 be the solution to the ODE*

$$\frac{\partial}{\partial t}\phi_1 = F(\phi_1),$$

with initial condition $\phi_1(0) = C_1$.

Then, $u(x, t) \geq \phi_1(t)$ for all $x \in M$, and for all t such that ϕ_1 exists.

Analogously, let u be a subsolution to (4.8). Assume that there exists $C_2 > 0$ such that $u(x, 0) \leq C_2$ for all x in M , and let ϕ_2 be the solution to the ODE

$$\frac{\partial}{\partial t}\phi_2 = F(\phi_2),$$

with initial condition $\phi_2(0) = C_2$.

Then, $u(x, t) \leq \phi_2(t)$ for all $x \in M$, and for all t such that ϕ_2 exists.

Proof. We will only proof the first part of the theorem, as the proof of the second part is analogous. For simplicity, let $\phi_1 = \phi$, $C_1 = C$. We have

$$\frac{\partial}{\partial t}(u - \phi) \geq \Delta(u - \phi) + F(u) - F(\phi),$$

and $(u - \phi)|_{t=0} \geq 0$.

Claim: $u - \phi \geq 0$ for all times where ϕ is defined.

In order to prove the claim, fix $\tau \in (0, T)$. Since M is compact, there exists K_τ such that both $|u(x, t)|$ and $|\phi(t)|$ are bounded above by K_τ , for all $x \in M$, for all $t \in [0, \tau]$.

Let L_τ be the Lipschitz constant for F in the τ -slice. Then

$$\frac{\partial}{\partial t}(u - \phi) \geq \Delta(u - \phi) \pm L_\tau(u - \phi).$$

Define $J = e^{K_\tau t}(u - \phi)$. An easy computation gives

$$\frac{\partial}{\partial t}J \geq \Delta J.$$

Since $J(0) \geq 0$, the result follows from the maximum principle for the heat equation, as follows.

We write $H = J + \varepsilon t + \varepsilon$ for some $\varepsilon > 0$. The heat equation for J implies that

$$\frac{\partial}{\partial t}H \geq \Delta H + \varepsilon.$$

From our old Calculus lessons, we remember that, if (x_0, t_0) is the point where the minimum of H is attained, *i.e.*,

$$H(x_0, t_0) = \min\{H(x, t); x \in M, t \in [0, t_0]\},$$

then

$$\frac{\partial}{\partial t}H(x_0, t_0) \leq 0; \quad \nabla H(x_0, t_0) = 0; \quad \Delta H(x_0, t_0) \geq 0. \quad (4.9)$$

Now, suppose by contradiction that our $H = J + \varepsilon t + \varepsilon$ is non-positive at some point (x_1, t_1) . Since $H(\cdot, 0) > 0$, then there exist a first time t_0 and a point x_0 such that $H(x_0, t_0) = 0$, which implies that $J(x_0, t_0) < 0$.

Recall that $H(x_0, t_0) = \min\{H(x, t); x \in M, t \in [0, t_0]\}$. Therefore, (4.9) yields

$$0 \geq \frac{\partial}{\partial t} H(x_0, t_0) \geq \Delta H(x_0, t_0) + \varepsilon \geq \varepsilon > 0,$$

which is a contradiction.

This completes the proof of the theorem. \square

4.1.2 Back to the evolution of R

As a direct application of the maximum principle to (4.7), we see that, if at $t = 0$ we have $R \geq 0$, then this relation holds for all times where the solution exists. Similarly, the condition $R \leq 0$ is also preserved for the normalized Ricci flow on surfaces. An important observation is that for higher dimensions, these quantities are not preserved under Ricci flow.

In fact, if $R(0) \leq 0$, the maximum principle tells us even more.

Proposition 4.1.1. *If there exist $C, \varepsilon > 0$ such that $-C \leq R(0) \leq -\varepsilon < 0$, then*

$$re^{\varepsilon t} \leq r - R \leq Ce^{\varepsilon t}$$

so R approaches r exponentially.

Proof. For a fixed t , let $\rho(t) = \min_{x \in M} R(x, t)$ and $\varrho(t) = \max_{x \in M} R(x, t)$. Then, ϱ satisfies

$$\frac{\partial}{\partial t} \varrho \leq \varrho(\varrho - r) \leq -\varepsilon(\varrho - r)$$

and the minimum ρ satisfies

$$\frac{\partial}{\partial t} \rho \geq \rho(\rho - r) \geq r(\rho - r),$$

which clearly imply the claim. \square

Corollary 4.1.1. *On a compact surface, if we start the Ricci flow with an initial metric $g(0)$ whose scalar curvature is negative, then the solution exists for all times, and converges exponentially to a metric of constant curvature.*

This is almost the result we seek in the case of a surface with $\chi(M) < 0$. The only remaining piece is to show that if $r < 0$, then the scalar curvature $R(t)$ will eventually become negative at some time. If that were true, we just restart the flow at that time, and the metric will flow exponentially to the metric with curvature equal to $r = \frac{4\pi\chi(M)}{A}$. We will be back to that soon.

Sadly, the situation is far more complicated for $\chi(M) > 0$. But we still have uniform lower bounds that are also consequence of the maximum principle.

Proposition 4.1.2. *Let $\rho(t) = \inf_{x \in M} R(x, t)$. Then*

- If $r > 0$, then

$$R - r \geq \frac{r}{1 - (1 - \frac{r}{\rho(0)})e^{rt}} - r \geq (\rho(0) - r)e^{rt}.$$

- If $r = 0$, then

$$R - r \geq \frac{\rho(0)}{1 - \rho(0)t}.$$

- If $r < 0$ and $\rho(0) < 0$, then

$$R \geq \frac{r}{1 - (1 - \frac{r}{\rho(0)})e^{rt}} \geq \rho(0)e^{-rt}.$$

This proposition follows from the maximum principle, that allows us to compare the solution of (4.7) with the solution to the associated scalar ODE:

$$\frac{\partial}{\partial t} R = R(r - R).$$

It seems that we juiced out all we could from the maximum principle for this equation. In order to obtain nice upper bounds for R evolving under Ricci flow, we need to use a smarter tool.

4.2 Ricci Solitons on surfaces

In this section, a new tool is developed in order to produce the expected upper bounds.

Definition 4.2.1. Let $g(t)$ be a solution of the normalized Ricci flow (4.2) on a surface M . We say that $g(t)$ is a **self-similar solution** if there exists a 1-parameter family $\varphi(t)$ of conformal diffeomorphisms such that

$$g(t) = [\varphi(t)]^* g(0). \quad (4.10)$$

Equation (4.10) implies that

$$\frac{\partial}{\partial t} g(t) = \mathcal{L}_X g,$$

where X is the vector field generated by $\varphi(t)$ and \mathcal{L}_X denotes the Lie derivative of the metric in the direction of X . More is true: if $g(t)$ is a solution to the normalized Ricci flow, then

$$(R - r)g_{ij} = \nabla_i X_j + \nabla_j X_i. \quad (4.11)$$

We shall refer to (4.11) as the **Ricci soliton equation**.

Also, if there exists a function f such that $\nabla f = X$, then we have a so-called **gradient Ricci soliton**, which will satisfy the equation

$$(R - r)g_{ij} = \nabla_i \nabla_j f. \quad (4.12)$$

The function f will be referred to as the **Ricci potential**.

Tracing the equation above, we see that the potential f must satisfy

$$\Delta f = R - r.$$

This equation is solvable on a compact manifold, since

$$\int_M (R - r) d\text{Vol} = 0.$$

A gradient Ricci soliton is a very special solution of the normalized Ricci flow, and as so, we expect that some quantities related to it will be preserved along the flow.

Lemma 4.2.1. For a gradient Ricci soliton, the expression

$$R + |\nabla f|^2 + rf$$

is only a function of time.

Proof. Let $M = \nabla\nabla f - 1/2\Delta f g$ be the trace-free part of the hessian of f . Clearly, $M = 0$ is a necessary and sufficient condition for a gradient Ricci soliton.

Computing the divergence of M , we have

$$(\operatorname{div} M)_i = \nabla^j M_{ij} = \frac{1}{2}(R\nabla_i f + \nabla_i R).$$

Therefore, for a gradient Ricci soliton,

$$\begin{aligned} 0 &= \nabla_i R + R\nabla_i f = \nabla_i R + (R - r)\nabla_i f + r\nabla_i f = \\ &= \nabla_i(R + |\nabla f|^2 + rf), \end{aligned} \quad (4.13)$$

which completes the proof. \square

In fact, we can choose our potential to satisfy an even nicer equation.

Lemma 4.2.2. *Let f_0 be a potential. Then, there exists $c = c(t)$ (that only depends on time) such that the new potential $f = f_0 + c$ satisfies*

$$\frac{\partial}{\partial t} f = \Delta f + rf. \quad (4.14)$$

The proof of this lemma follows from noticing that $\frac{\partial}{\partial t} \Delta = (R - r)\Delta$, and recalling that the only harmonic functions on a compact surface are constants.

Applying the maximum principle to (4.14), we see that there exists a constant K such that $|f| \leq C^{rt}$. We still need to work a bit further to extract an upper bound for R . For that, we define

$$h = \Delta f + |\nabla f|^2 = (R - r)f + |\nabla f|^2. \quad (4.15)$$

Lemma 4.2.3. *The evolution for h is given by*

$$\frac{\partial}{\partial t} h = \Delta h - 2|M|^2 + rh,$$

where M is the trace-free part of the Hessian of f .

Therefore, if $h \leq C$ at time zero, then $h(t) \leq C^{rt}$ for all t .

Proof. From the evolution of R , we have

$$\frac{\partial}{\partial t} R = \Delta(R - r) + R(r - R) = \Delta(R - r) + (\Delta f)^2 + r(R - r).$$

Also,

$$\frac{\partial}{\partial t} |\nabla f|^2 = \frac{\partial}{\partial t} (g^{ij} \nabla_i \nabla_j f) = \Delta(|\nabla f|^2) - 2|\nabla \nabla f|^2 + r|\nabla f|^2,$$

which implies the claim. \square

Observing that $R = h + r - |\nabla f|^2$, we obtain our desired upper bound:

Theorem 4.2.1. (Hamilton) *For any initial metric on a compact surface, there exists a constant C such that*

$$-C \leq R \leq Ce^{rt} + r.$$

Therefore, the Ricci flow equation has a solution defined for all times.

For the $r < 0$ case, we complete the proof of the main result of this chapter, as the theorem above implies that eventually R will become negative. Combining this with Corollary 4.1.1, we obtain

Corollary 4.2.1. *If $r < 0$ (a purely topological condition), then for any choice of initial metric, the solution to the Ricci flow exists for all times, and converges to a metric with constant negative curvature.*

4.3 The case $r = 0$

From Theorem 4.2.1, we know that the solution exists for all times, and that R is bounded from above and below. It remains to show that the solution actually converges to a flat metric.

Recall that the Ricci flow on surfaces evolves inside a conformal class. So, consider $g = e^{-2u}h$, two conformal metrics. The relation between the scalar curvatures is given by

$$R_g = e^{2u}(R_h - 2\Delta_h u).$$

Hence, up to replacing the starting metric by a conformal factor u such that $\Delta u = R^2$, we may assume that h is the flat metric, and

²This equation is solvable, as the average scalar curvature is zero

study the evolution of the conformal factor u .

Note that

$$-2\frac{\partial}{\partial t}ue^{-2u}h = \frac{\partial}{\partial t}g = R_g g = R_g e^{-2u}h = -2\Delta_h u h,$$

which shows that the conformal factor u evolves by

$$\frac{\partial}{\partial t}u = e^{2u}\Delta_h u. \quad (4.16)$$

The maximum principle allows us to conclude the following.

Corollary 4.3.1. *There exists a constant $C \geq 0$ such that*

$$-C \leq u(t) \leq C$$

for all time t .

This corollary has important consequences: all the metrics $g(t)$ are uniformly equivalent, as well as the diameter and Sobolev constant.

From now on, our goal is to prove exponential decay for the scalar curvature $R(t)$. The strategy, following [Ham88], is estimate the L^2 -norms of R , ∇R and $\nabla^2 R$, and use the Sobolev embedding theorem to obtain the expected decay.

In what follows, we will drop the subscript h on the expressions for the geometric objects related to the (flat) metric h . We have

$$\frac{d}{dt} \int |\nabla u|^2 d\mu = 2 \int \langle \nabla u, \nabla(e^{2u} \Delta u) \rangle d\mu = -2 \int e^{2u} \Delta u d\mu, \quad (4.17)$$

which yields, together with the relation

$$\begin{aligned} \int (\Delta u)^2 d\mu &\geq c \int |\nabla u|^2 d\mu, \\ \frac{d}{dt} \int |\nabla u|^2 d\mu + c \int |\nabla u|^2 d\mu &\leq 0. \end{aligned}$$

Thinking of this integral as a function $A(t)$, the relation above simply tells us that $\frac{d}{dt}A \leq -cA$. This implies the exponential decay on the L^2 -norm of ∇u : for some $C > 0$, we have

$$\int (\nabla u)^2 d\mu \geq C e^{-ct}.$$

Now, integrating (4.17) with respect to t , we get

$$\int_T^\infty \int e^{2u} (\Delta u)^2 d\mu \leq \int (\nabla u)^2 d\mu,$$

and since u is uniformly bounded,

$$\int_T^\infty \int R^2 d\mu \leq \int_T^\infty \int e^{2u} (\Delta u)^2 d\mu \leq C e^{-ct}.$$

This relation tells us that there exist a point ξ in every interval $[T, T+1]$ such that

$$\int R^2 d\mu \leq C e^{-c\xi}.$$

Also, from the evolution equation of R ,

$$\frac{\partial}{\partial t} R^2 = 2R(R^2 + \Delta R),$$

and hence

$$\frac{d}{dt} \int R^2 d\mu \leq \int R^3 d\mu \leq C \int R^2 d\mu, \quad (4.18)$$

where the last inequality comes from the fact that R is bounded and the surface is compact.

Let $q(t) = \int R^2 d\mu$. We learned so far that in every interval $[T, T+1]$, $q(\xi) \leq C e^{-c\xi}$, and that $\frac{d}{dt} q \leq C q$. Therefore, for $t \in [T, T+1]$

$$q(t) = q(\xi) + \int_\xi^t \frac{d}{dt} q \leq C e^{-c\xi} + C \int_T^\infty q \leq C e^{-ct}, \quad (4.19)$$

which shows the exponential decay

$$\int R^2 d\mu \leq C e^{-ct}.$$

Now, we proceed with the estimates for the gradient of R . Integrating (4.18) with respect to time, and noting that any L^p -norm of R goes to zero exponentially, we get

$$\int_T^\infty \int |\nabla R|^2 d\mu \leq C e^{-cT},$$

which again shows that

$$\int |\nabla R|^2 d\mu \leq C e^{-c\xi}$$

for some ξ in every interval $[T, T+1]$.

Integration by parts yields

$$\frac{d}{dt} \int |\nabla R|^2 d\mu + 2 \int (\Delta R)^2 d\mu \leq -2 \int R^2 \Delta R d\mu,$$

and Cauchy-Schwartz gives that the right-hand side is bounded by

$$-2 \int R^2 \Delta R d\mu \leq \int R^4 d\mu + \int (\Delta R)^2 d\mu.$$

Therefore,

$$\frac{d}{dt} \int |\nabla R|^2 d\mu + \int (\Delta R)^2 d\mu \leq \int R^4 d\mu$$

Since the right-hand side of the equation above is exponentially small, this gives us two pieces of information: firstly,

$$\frac{d}{dt} \int |\nabla R|^2 d\mu \leq C e^{-ct},$$

and secondly,

$$\int_T^\infty \int (\Delta R)^2 d\mu \leq C e^{-cT}.$$

This last inequality tells us that we can play the same game as before, redefining the quantity q to be $q(t) = \int (\Delta R)^2 d\mu$, and obtaining the desired exponential decay on the L^2 -norm of the Laplacian of R .

The bound on $\nabla^2 R$ follows from the previous bounds, and the **Bochner identity** for the case of a flat metric

$$\Delta\left(\frac{1}{2}|\nabla^2 R|^2\right) = |\nabla^2 R|^2 + \langle \nabla R, \nabla(\Delta R) \rangle.$$

imply that³

$$\int |\nabla^2 R|^2 d\mu = \int (\Delta R)^2 d\mu - \frac{1}{2} \int R |\nabla R|^2 d\mu.$$

³Note that also an integration by parts is needed.

Therefore, with the L^2 -norms of R , ∇R and $\nabla^2 R$ in hand, Sobolev's embedding tells us that the maximum of $|R|$ goes to zero exponentially, which completes our proof of the Uniformization Theorem in the case $\chi(M) = 0$.

4.4 The case $R > 0$

4.4.1 Hamilton's Harnack inequality

Following the seminal paper [Ham88], we begin by deriving a generalization, by Hamilton, of the Li-Yau Harnack inequality (cf. [LY86]). For completeness, we state here the classical Harnack inequality.

Theorem 4.4.1. *Let M be a compact manifold of dimension n with a fixed metric with non-negative Ricci curvature. Let $f > 0$ be a solution of*

$$\frac{\partial}{\partial t} f = \Delta f$$

on $(0, T)$. Then, for any two points (ξ, τ) and (X, T) in space-time with $0 < \tau < T$,

$$\tau^{n/2} f(\xi, \tau) \leq e^{\Delta/4 T^{n/2}} f(X, T),$$

where $\Delta = \frac{d^2(\xi, X)}{T - \tau}$, and d is the distance along the shortest geodesic.

The idea of the proof, in very rough words, is to study the evolution equation of the quantity $L = \log f$, and apply the maximum principle for $Q = \Delta L$.

For the case of the Ricci flow on surfaces, we need a better version of the Harnack inequality, as the metrics on the manifold are varying. Hamilton's idea is to consider a new definition for Δ , as follows.

Definition 4.4.1. *Let $g(t)$ a family of metrics on a manifold M . Define*

$$\Delta((\xi, \tau)(X, T)) = \inf_{\gamma} \int_{\tau}^T \frac{ds^2}{dt} dt, \quad (4.20)$$

where the infimum is taken over all paths joining (ξ, τ) to (X, T) .

Note that this definition coincides with the previous one in the case of a fixed metric. Also, if there are two metrics h and G , independent of time, with distances $d_h(\xi, X)$ and $d_G(\xi, X)$, then

$$\frac{d_h^2(\xi, X)}{T-t} \leq \Delta((\xi, \tau)(X, T)) \leq \frac{d_G^2(\xi, X)}{T-t}$$

whenever $h(x) \leq g(x, t) \leq G(x)$.

Now, we state Hamilton's Li-Yau Harnack inequality.

Theorem 4.4.2. *Let g be a solution of Ricci flow on a compact surface, with $R > 0$ for $0 < \tau < T$. Then, for any two points (ξ, τ) and (X, T) in space-time with $0 < \tau < T$,*

$$(e^{r\tau} - 1)R(\xi, \tau) \leq e^{\Delta/4}(e^{rT} - 1)R(X, T),$$

where Δ is as in Definition 4.4.1.

Proof. Let $L = \log R$. Then,

$$\frac{\partial}{\partial t}L = \Delta L + |\nabla R|^2 + R - r.$$

Consider the quantity $Q = \frac{\partial}{\partial t}L - |\nabla R|^2$. The evolution equation for Q is given by

$$\begin{aligned} \frac{\partial}{\partial t}Q &= \Delta \left(\frac{\partial}{\partial t}L \right) + \frac{\partial}{\partial t}(\Delta)L + \frac{\partial}{\partial t}R = \\ &= \Delta(Q) + \Delta(|\nabla L|^2) + (R - r)\Delta L + R\frac{\partial}{\partial t}L = \\ &= \Delta(Q) + 2(|\nabla^2 L| + \langle \Delta \nabla L, \nabla L \rangle) + (R - r)\Delta L + R(\Delta L + |\nabla R|^2 + R - r) = \\ &= \Delta Q + 2|\nabla \nabla L|^2 + \langle \nabla \Delta L, \nabla L \rangle + 2R|\nabla L|^2 + (2R - r)\Delta L + R(R - r) = \\ &= \Delta Q + 2\langle \nabla L, \nabla Q \rangle + 2|\nabla \nabla L|^2 - \frac{1}{2}(R - r)g^2 + rQ \\ &\geq \Delta Q + 2\langle \nabla L, \nabla Q \rangle + Q^2 + rQ, \quad (4.21) \end{aligned}$$

where the inequality follows from

$$Q^2 \leq 2|\nabla \nabla L|^2 - \frac{1}{2}(R - r)g^2.$$

Once again, the maximum principle applied for Q allows us to compare Q with the solution of the associated ODE, giving

$$Q \leq \frac{-re^{rt}}{e^{rt} - 1}.$$

Now, let γ be any path joining two points (ξ, τ) and (X, T) in space-time. Now, we just compute

$$\begin{aligned} L(X, T) - L(\xi, \tau) &= \int_{\tau}^T \frac{d}{dt} L dt \\ &\geq \int_{\tau}^T \left[|\nabla L|^2 - \frac{-re^{rt}}{e^{rt} - 1} + \frac{\partial L}{\partial s} \frac{ds}{dt} \right] dt \\ &\geq -\log \left(\frac{e^{rT-1}}{e^{r\tau-1}} - \frac{1}{4} \int_{\tau}^T \frac{ds^2}{dt} dt \right). \end{aligned}$$

The proof follows from taking exponentials, and noting that the infimum of the last integral over all paths is the definition of Δ . \square

4.4.2 Entropy estimate

Another step in developing the behavior at infinite time of R is the following result.

Theorem 4.4.3. *Let R be the scalar curvature of the solution of the Ricci flow on a surface, with $R > 0$. Then,*

$$\int R \log R$$

is decreasing, as a function of time.

Proof. Following Hamilton, we consider

$$Z = \frac{\int Q R d\mu}{\int R d\mu}.$$

Then, Z satisfies

$$\frac{dZ}{dt} \geq Z^2 + rZ.$$

If Z would become positive, it would blow up at a finite time, contradicting the long-time existence result for the Ricci flow on surfaces. Therefore, $Z \leq 0$.

Also,

$$Q = \frac{\Delta R}{R} - \frac{|\nabla R|^2}{R^2} + (R - r).$$

Then, if $R > 0$, we get that

$$\int (R - r)^2 d\mu \leq \int \frac{|\nabla R|^2}{R}.$$

To complete the proof, just observe that

$$\begin{aligned} \frac{d}{dt} \int R \log R d\mu &= \int \left[\frac{dR}{dt} \log R d\mu + \frac{dR}{dt} + R(R - r) \log R \right] d\mu = \\ &= \int [\Delta R \log R + \Delta R + R(r - R)] d\mu = \\ &= \int (R - r)^2 d\mu - \int \frac{|\nabla R|^2}{R} d\mu \leq 0. \end{aligned} \quad (4.22)$$

□

4.4.3 Uniform bounds for R .

Now, we want to combine Hamilton's Li-Yau harnack inequality with the entropy estimates to obtain uniform bounds for R .

Let $R_{\min}(t) = \min_{x \in M} R(x, t)$ and $R_{\max}(t) = \max_{x \in M} R(x, t)$.

Claim: For any $t \in [\tau, \tau + (2R_{\max}(\tau))] = [\tau, T]$,

$$R_{\max}(t) \leq 2R_{\max}(\tau).$$

To see this, recall that the evolution equation for the curvature is given by

$$\frac{\partial}{\partial t} R = \Delta R + R^2 - rR.$$

Hence, at a maximum in space (where $R_{\max} > 0$),

$$\frac{\partial}{\partial t} R \leq \Delta R + R^2.$$

By the maximum principle, we can compare R with the solution of the associated ODE, obtaining

$$R_{\max}(t) \leq \frac{1}{R_{\max}^{-1} + \tau - t} \leq 2R_{\max}(\tau).$$

This allows us to conclude that

$$g(x, \tau) \leq eg(x, t)$$

for any t in the interval $[\tau, \tau + (2R_{\max}(\tau))]$. This follows from integrating the Ricci flow equation:

$$g(x, t) = e^K g(x, \tau),$$

where

$$K = \int_{\tau}^t (r - R) ds \geq \int_{\tau}^t -R ds \geq -2 \int_{\tau}^T R_{\max}(\tau) ds = -1.$$

Hence, if d is the geodesic distance at time T ,

$$\Delta(\xi, \tau, X, T) \leq C \frac{d^2(X, \xi)}{T - t}.$$

Applying Hamilton's Li-Yau Harnack inequality, and noting that $\frac{e^{r\tau}-1}{e^{rT}-1}$ is a topological constant,

$$R(\xi, \tau) \leq CR(X, T) \quad \text{for all } x \in B\rho(\xi),$$

where⁴

$$\rho = \frac{\pi}{2R_{\max}(T)^2}.$$

On the other hand, using the entropy estimate, we obtain

$$\begin{aligned} C &\geq \int R \log R d\mu \geq \int_{B_\rho} R \log(cR_{\max}(\tau)) d\mu \\ &\geq c \int_{B_\rho} R d\mu + \log(cR_{\max}(\tau)) c \int_{B_\rho} R d\mu \geq C \log(cR_{\max}(T)), \end{aligned} \quad (4.23)$$

⁴This is an application of Klingenberg's Theorem.

i.e., $R_{\max}(T)$ is bounded, hence $R_{\max}(\tau)$ is bounded. Since this is true for all $\tau > 1$, then R is bounded.

Recall that a bound on R gives a lower bound on the injectivity radius, and hence an upper bound on the diameter (since the area is constant for the normalized Ricci flow). Then, if $T - \tau \leq 1$,

$$\Delta(\xi, \tau, X, T) \leq \frac{C}{T - \tau}.$$

Combining with Harnack, we obtain, for $t \geq 1$,

$$R(x, t) \leq CR(y, t + 1)$$

for any two points x, y .

This completes the proof of the following result.

Theorem 4.4.4. *For a solution of the Ricci flow with $R > 0$ on a compact surface, there exist constants $0 < c < C < \infty$ such that, for all times,*

$$c \leq R(t) \leq C.$$

We will refer the reader to [Ham88] for the proof of the uniform bounds of the derivatives of R . The proof is to estimate inductively the L^2 -norms of the derivatives, and to observe that the Sobolev constants may be taken uniformly.

4.4.4 Convergence to a constant curvature metric

The strategy to complete the proof of the main result of this chapter, that is, to prove that the Ricci flow on surfaces converge to a metric with constant curvature, is to modify a bit the flow equation by a innocuous term, and note that the resulting flow converges to a desired metric.

Recall the definition of the trace-free Hessian of f

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2} \Delta f g_{ij},$$

where f is the potential for the curvature, which satisfies

$$\Delta f = R - r.$$

Lemma 4.4.1. *The evolution equation for M is given by*

$$\frac{\partial}{\partial t}|M_{ij}|^2 = \Delta|M_{ij}|^2 - 2|\nabla_k M_{ij}|^2 - 2R|M_{ij}|^2.$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t}M_{ij} &= \nabla_i \nabla_j \frac{\partial}{\partial t}f - \left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k f - \frac{1}{2}\frac{\partial}{\partial t}[(R-r)g_{ij}] \\ &= \nabla_i \nabla_j \Delta f + \frac{1}{2}(\nabla_i R \nabla_j f + \nabla_j R \nabla_i f - \langle \nabla R, \nabla f \rangle g_{ij}) \\ &\quad - \frac{1}{2}\Delta R g_{ij} + r M_{ij}. \end{aligned}$$

Recalling the formula for the commutator of Δ and $\nabla \nabla$, we obtain

$$\frac{\partial}{\partial t}M_{ij} = \Delta \nabla_i \nabla_j f - 2R M_{ij} - \frac{1}{2}\Delta R g_{ij} + r M_{ij}.$$

□

Once again, we apply the maximum principle to get

Corollary 4.4.1. *If $R \geq c > 0$, then*

$$|M_{ij}| \leq C e^{ct}.$$

Now, we consider the modified flow

$$\frac{\partial}{\partial t}g_{ij} = M_{ij} = (r - R)g_{ij} - 2\nabla_i \nabla_j f,$$

which differs from the Ricci flow only by transport along a family of diffeomorphisms generated by the gradient vector field of f . Note that $|M_{ij}|$ is invariant under diffeomorphisms.

We will refer to [CK04] or [Ham88] for the proof that also the derivatives of M_{ij} decay exponentially to zero.

So, we proved that the modified Ricci flow converges exponentially to a metric g_∞ such that the corresponding M_∞ vanishes. Therefore, g_∞ is a gradient Ricci soliton. If we prove that the only gradient Ricci solitons on a compact surface are the trivial ones (metrics with constant curvature), the diffeomorphism invariance tells us that the solution to the normalized Ricci flow on a compact surface with $R > 0$ converges to a metric of constant curvature.

Proposition 4.4.1. *If $g(t)$ is a soliton solution to normalized Ricci flow on S^2 , then $g(t) = g(0)$ is a metric with constant curvature.*

Proof. The equation for a Ricci soliton is given by

$$(r - R)g_{ij} = \nabla_i X + \nabla_j X.$$

Contracting with Rg^{-1} , we get

$$2R(r - R) = 2R\operatorname{div} X,$$

and hence

$$-\int_{S^2} (R - r)^2 d\mu = \int_{S^2} R\operatorname{div} X d\mu.$$

Now, using the Kazdan-Warner identity, we obtain

$$\int_{S^2} (R - r)^2 d\mu = \int_{S^2} \langle \nabla R, X \rangle d\mu = 0,$$

and hence $R = r$. □

An important remark is that the Kazdan-Warner identity is the only place where we assumed the Uniformization Theorem for the sphere. In fact, Chen, Lu and Tian [CLT06] provide a proof that the only gradient shrinking solitons on a sphere are the trivial ones without using uniformization. Therefore, the Ricci flow can be used to prove the Uniformization Theorem.

4.5 The case where $r > 0$, but R is of mixed sign.

In [Cho91], Chow completes the proof of the main result of this chapter, by proving the following theorem.

Theorem 4.5.1. *If g is any initial metric in S^2 , then under the normalized Ricci flow, the scalar curvature R becomes positive in finite time.*

Once the curvature R becomes positive, we may "restart" the flow, and the results on the previous section will guarantee the convergence to a constant curvature metric.

Chow's idea is to prove a modified Hamilton-Li-Yau Harnack inequality for the case where R is of mixed sign, and combine the flow with certain quantities that help to control the negative parts of the curvature. We refer the reader to [Cho91] for the complete, well-detailed exposition of the result.

Chapter 5

Short-Time Existence

As we have seen in Chapter 1, the Ricci flow is not parabolic, so standard PDE techniques are not sufficient to guarantee that (1.2) has a solution even for short time.

Hamilton, in [Ham88], first proved the short-time existence by the use of a very fancy analytic tool, the Nash-Moser implicit function theorem, and his proof was very deep and intruncated. Soon after, DeTurck [DeT83] provided a much simpler proof, by noting that the Ricci flow equation can be related to a modified parabolic system, in the sense that solutions of the modified equation will be related to the original Ricci flow. That is the subject of this chapter.

5.1 The linearization of the Ricci tensor

We start by recalling a few definitions from the theory of partial differential equations. In this section, the reader should be warned that the multi-index notation is being used.

Let \mathcal{V} and \mathcal{W} be vector bundles over M , and let L be a **differential operator of order k** on M , that can be written as

$$L(V) = \sum_{|\alpha| \leq k} L_\alpha \partial^\alpha V,$$

for $V \in C^\infty(\mathcal{V})$, where $L_\alpha \in \text{Hom}(\mathcal{V}, \mathcal{W})$.

Definition 5.1.1. We define the **principal symbol** of the linear differential operator L in the direction of the covector ξ by

$$\sigma(L)(\xi) = \sum_{|\alpha|=k} L_\alpha(\Pi_j \xi^{\alpha_j}).$$

We leave an easy exercise for the reader: what is the principal symbol for the Laplacian Δ in \mathbb{R}^n ?

We now regard $\text{Ric}(g)$ as a non-linear partial differential operator on the metric g :

$$\text{Ric}_g = \text{Ric}(g) : C^\infty(S_2^+ T^* M) \rightarrow C^\infty(S_2 T^* M).$$

Its linearization is given by

$$[\text{Ric}'_g(h)]_{jk} = \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{pq}).$$

The principal symbol $\sigma(\text{Ric}'_g)$ in the direction of ξ is given by replacing ∇_i by ξ_i :

$$[\sigma(\text{Ric}'_g)(\xi)(h)]_{jk} = \frac{1}{2} g^{pq} (\xi_q \xi_j h_{kp} + \xi_q \xi_k h_{jp} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{pq}).$$

Definition 5.1.2. A linear partial differential operator L is said to be **elliptic** if the principal symbol $\sigma(L)(\xi)$ is an isomorphism for any $\xi \neq 0$.

A non-linear partial differential operator N is **elliptic** if its linearization N' is elliptic.

Unfortunately, due to the invariance of the Ricci tensor with respect to diffeomorphism, i.e.,

$$\text{Ric}(\phi^*(g)) = \phi^*(\text{Ric}(g)), \quad (5.1)$$

we will see that the principal symbol of Ric has non-trivial kernel.

Following the notation in [DeT83], we define, for any symmetric tensor $T \in S^2(T^* M)$,

$$\begin{aligned} \text{tr}(T) &= g^{kl} T_{kl}; \\ G(T)_{ij} &= T_{ij} - \frac{1}{2} (\text{tr}(T)) g_{ij}; \\ \delta(T)_i &= -g^{jk} \nabla_k T_{ij} \end{aligned}$$

As we see, the **divergence δ of a 2-tensor** defines a map from symmetric 2-tensors to 1-forms. So, we can define an L^2 -adjoint $\delta^* : T^*M \rightarrow S^2T^*M$ by

$$v \mapsto \delta^*(v) = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j) = \mathcal{L}_{v^\#} g.$$

The total symbol of δ_g^* in the direction of ξ is given by

$$[\sigma[\delta_g^*](\xi)X]_{ij} = \frac{1}{2}(\xi_i X_j + \xi_j X_i).$$

In order to show that the principal symbol of Ric is not an isomorphism, consider the composition

$$\text{Ric}'_g \circ \delta_g^* : C^\infty(T^*M) \rightarrow C^\infty(S^2T^*M),$$

which is, *a priori*, a third-order partial differential operator. So, its principal symbol should have degree 3. However, due to (5.1), we have

$$[\text{Ric}'_g \circ \delta_g^*(X)]_{ij} = \frac{1}{2} [\mathcal{L}_{X^\#}(\text{Ric}_g)]_{ij},$$

where $\mathcal{L}_{X^\#}$ is the Lie derivative in the direction of the vector $X^\#$, dual to X .

Note that the right-hand side of the equation above only involves one derivative of X , hence its principal symbol is at most of degree 1 in ξ . Therefore, we conclude that the principal symbol of $\text{Ric}'_g \circ \delta_g^*$ is the zero map.

A property of the symbol, whose simple proof is left to the reader, is that the symbol of a composition of two operators is the composition of the symbols of the operators. Then,

$$\begin{aligned} 0 &= \sigma[\text{Ric}'_g \circ \delta_g^*](\xi) \\ &= \sigma[\text{Ric}'_g](\xi) \circ \sigma[\delta_g^*](\xi) \end{aligned}$$

Therefore, the image of δ_g^* is in the kernel of the linearized Ricci operator. So,

$$\dim \text{Ker} \sigma[\text{Ric}'_g](\xi) \geq n,$$

showing that the Ricci operator is not elliptic.

In fact, we refer to [CK04] for the proof that the dimension of that kernel is exactly n , which tells us that the failure of the Ricci operator to being elliptic relies solely on its diffeomorphism invariance.

DeTurck, in [DeT83], made a very clever use of this fact: he noticed that, by modifying the Ricci flow equation by adding a term which is the Lie derivative of a metric with respect to a vector field, that in its turn depends on the metric.

The modified equation becomes parabolic, and its solution can be pulled back to a solution to the original Ricci flow equation by a carefully chosen family of diffeomorphisms. This is the subject of next section.

5.2 DeTurck trick

Rewriting the linearization of the Ric operator,

$$\text{Ric}'(g)(h) = \frac{1}{2}\Delta_L h - \delta^*(\delta G(h)), \quad (5.2)$$

where the first term is the **Lichnerovicz Laplacian** for tensors, given by

$$(\Delta_L)_{jk} = \Delta h_{jk} + 2g^{qp}R_{qjk}^r h_{rp} - g^{qp}R_{jp}h_{qk} - g^{qp}R_{kp}h_{jq}. \quad (5.3)$$

and the second term in (5.2) is such that symmetric squares of 1-forms belong to the kernel of the linearized Ricci operator.

DeTurck, in [DeT82], showed that for any choice of symmetric 2-tensor T , the expression $\delta^*(T^{-1}\delta G(T))$ has always the same symbol as the second term in the right-hand side of (5.2). With this in mind, it is natural to consider the modified operator

$$Q(g) = \text{Ric}(g) - \delta^*(T^{-1}\delta G(T)),$$

which is elliptic! The obvious choice for a symmetric tensor T is to take the initial Riemannian metric g_0 on M .

Therefore, the parabolic system

$$\left\{ \frac{\partial}{\partial t} g = -2Q(g) \right. \quad (5.4)$$

has a solution defined for short time.

Now, we need to argue that a solution of (5.4) can be translated to a solution of the Ricci flow equation (1.2).

Lemma 5.2.1. *Let $v(y, t)$ be a vector field on M . Then, for small t , there exists a unique family of diffeomorphisms $\phi_t : M \rightarrow M$ such that*

$$\begin{cases} \frac{\partial}{\partial t} \phi_t(x) = v(\phi_t(x), t), \\ \phi_0(x) = id. \end{cases}$$

The proof of this lemma is left as an exercise to the reader, as it is analogous to the standard ODE case.

Another necessary ingredient is the following result.

Lemma 5.2.2. *Let $g(t)$ be a family of Riemannian metrics, and let ϕ_t be the 1-parameter diffeomorphism family related to the vector field $v(y, t)$. Then*

$$\frac{\partial}{\partial t} \phi_t^*(g)(x) = \phi_t^* \left(\frac{\partial}{\partial t} g(\phi_t(x)) \right) + 2\phi_t^* (\delta^* v^\sharp(\phi_t(x))), \quad (5.5)$$

where \sharp is the map corresponding vector fields and 1-forms. Both the maps \sharp and δ^* are defined with respect to g .

Now we are in position of proving the main result of this section. Choose v to be the vector field dual to the 1-form

$$v^\sharp = -T^{-1}(\delta G(T)),$$

and let ϕ_t be the 1-parameter family of diffeomorphisms associated to v .

If g is a solution to the modified equation (5.4), we have

$$\begin{aligned} \frac{\partial}{\partial t} \phi_t^*(g) &= \phi_t^* \left(\frac{\partial}{\partial t} g \right) + 2\phi_t^* (\delta^* v^\sharp) \\ &= \phi_t^* (-2Q(g)) + 2\phi_t^* (\delta^* v^\sharp) \\ &= -2\text{Ric}(\phi_t^*(g)), \end{aligned}$$

that is, $\phi_t^*(g)$ is a solution to (1.2).

This completes the proof of the short-time existence of a solution to the unnormalized Ricci flow .

Chapter 6

Ricci Flow in 3 dimensions

The goal of this chapter is to present the proof of Hamilton's theorem for 3-dimensional Riemannian manifolds with positive Ricci curvature, following the paper [Ham82].

Theorem 6.0.1. (Hamilton) *Let M be a compact 3-dimensional manifold which admits a metric with strictly positive Ricci curvature. Then, M admits a metric of constant positive curvature.*

In order to complete this task, we will study thoroughly the Ricci flow equation (1.2) on three-dimensional manifolds.

6.1 Evolution of curvatures under the Ricci flow

We state here the evolution equations for all the geometric objects under Ricci flow . For the proof of those expressions, we refer the reader to [CK04].

Lemma 6.1.1. *Let $g(t)$ be a solution to Ricci flow . Then,*

1. The Levi-Civita connection $\Gamma(g)$ evolves by

$$\frac{\partial}{\partial t} \Gamma_{ij}^k = -g^{kl} (\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}). \quad (6.1)$$

2. The Ricci tensor $\text{Ric}(g)$ evolves by

$$\frac{\partial}{\partial t} R_{jk} = \Delta R_{jk} + \nabla_j \nabla_k R - g^{pq} (\nabla_q \nabla_j R_{kp} + \nabla_q \nabla_k R_{jp}). \quad (6.2)$$

3. The scalar curvature R evolves by

$$\frac{\partial}{\partial t} R = 2\Delta R - 2g^{jk} g^{pq} \nabla_q \nabla_j R_{kp} + 2|\text{Ric}|^2. \quad (6.3)$$

From the second Bianchi identity

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0,$$

we obtain a nice form for the evolution of R :

$$\frac{\partial}{\partial t} R = \Delta R + 2|\text{Ric}|^2. \quad (6.4)$$

Note that $|\text{Ric}|^2 \geq 0$, hence the maximum principle applied to (6.4) implies the following positivity result:

Lemma 6.1.2. *Let $g(t)$ be a solution to Ricci flow with initial condition g_0 .*

If the scalar curvature $R(g_0)$ of the initial metric is bounded below by some constant C , then $R(g(t)) \geq C$ for as long as the solution exists.

Now we analyze the Ricci tensor. Its evolution is given by

$$\frac{\partial}{\partial t} \text{Ric}_{ij} = \Delta_L \text{Ric}_{jk},$$

where Δ_L is the Lichnerovitz laplacian defined in the previous chapter.

Written in this form, one may be led to think that the maximum principle could be applied to the equation above, and that would

imply that positivity of the Ricci curvature is preserved along the flow. However, Δ_L contains terms coming from the whole Riemann curvature tensor, so more work is need to guarantee that positivity is preserved.

We note that in 3 dimensions, the Weyl tensor¹ vanishes identically, which allows the Ricci curvature to determine completely the Riemann tensor:

$$R_{ijkl} = R_{il}g_{jk} + R_{jk}g_{il} - R_{ik}g_{jl} + R_{jl}g_{ik} - \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl}).$$

Hence, we can rewrite the evolution of the Ricci curvature as

$$\begin{aligned} \frac{\partial}{\partial t} \text{Ric}_{jk} &= \Delta R_{jk} + 3R R_{jk} - 6g^{pq} R_{jp} R_{qk} + (2|\text{Ric}|^2 - R^2)g_{jk} \\ &= \Delta R_{jk} + Q_{jk}, \end{aligned} \tag{6.6}$$

where the tensor Q_{jk} is defined by the expression above.

6.2 Maximum Principle for tensors

In order to check that $\text{Ric} \geq 0$ is preserved along the flow, we need a version of the maximum principle that can be applied to tensors.

Let u^k be a vector field, and let M_{ij} , g_{ij} and N_{ij} be symmetric tensors (that may depend on t) on a compact manifold M . Assume that $N_{ij} = p(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using g_{ij} .

Definition 6.2.1. *The tensor N_{ij} satisfy the null-eigenvector condition if for any nullvector v^i of M_{ij} ,*

$$N_{ij}v^i v^j \geq 0.$$

Theorem 6.2.1. (Maximum Principle for tensors) *Suppose that on $0 \leq t \leq T$,*

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij},$$

¹The Weyl tensor is the conformal part of the curvature tensor.

where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition.

Then, if $M_{ij} \geq 0$ at time zero, then it remains greater than or equal to zero for all $t \in [0, T]$.

Proof. The strategy of the proof is to show that M_{ij} is non-negative on a small interval $0 \leq t \leq \delta$, where δ is very small (but uniform). Splitting the whole intervals in sizes smaller than such δ will complete the proof of the result.

We define

$$\tilde{M}_{ij} = M_{ij} + \varepsilon(\delta + t)g_{ij}.$$

Claim: $\tilde{M}_{ij} > 0$ on $0 \leq t \leq \delta$, for every $\varepsilon > 0$.

Letting $\varepsilon \rightarrow 0$ proves the theorem.

To prove the claim, assume not. Then, there exists a first time θ , $0 < \theta \leq \delta$ such that \tilde{M}_{ij} acquires a null unit eigenvector v^i at some point $p \in M$.

Then, if $\tilde{N}_{ij} = p(\tilde{M}_{ij}, g_{ij})$, then $\tilde{N}_{ij}v^iv^j \geq 0$ at (p, θ) . Moreover,

$$|\tilde{N}_{ij} - N_{ij}| \leq C|\tilde{M}_{ij} - M_{ij}|,$$

where C only depends on $\max(|\tilde{M}_{ij}| + |M_{ij}|)$, because \tilde{N} and N are just polynomials. Furthermore, if we choose $\varepsilon, \delta < 1$, then the constant C only depends on $\max(|M_{ij}|)$.

We can extend the null eigenvector v^i to a vector field on a neighborhood of the point $p \in M$, such that $\nabla_j v^i = 0$ at p , and such that the vector field is not time-dependent.

Write $f = \tilde{M}_{ij}v^iv^j$. Then, by our construction, $f \geq 0$ for all $0 \leq t \leq \theta$ and all of M . Hence, since (p, θ) is a minimum,

$$\frac{\partial}{\partial t}f(p) < 0, \quad \nabla_k f = 0 \quad \text{and} \quad \Delta f \geq 0 \quad (6.7)$$

at (p, θ) .

Recalling that v is not time-dependent, and g is a solution of the Ricci flow ,

$$\frac{\partial}{\partial t}f = \left(\frac{\partial}{\partial t}M_{ij}\right)v^iv^j + \varepsilon - 2\varepsilon(\delta + t)R_{ij}v^iv^j \quad (6.8)$$

$$\geq \left(\frac{\partial}{\partial t}M_{ij}\right)v^iv^j + \frac{\varepsilon}{2}, \quad (6.9)$$

provided that $\delta \leq (8 \max |R_{ij}|)^{-1}$.

Also, at the point (p, θ) ,

$$\nabla_k f = \nabla_k M_{ij} v^i v^j \quad \text{and} \quad \Delta f = \Delta M_{ij} v^i v^j. \quad (6.10)$$

Plugging these in (6.8), we see that

$$0 > \frac{\partial}{\partial t} f \geq \left(\frac{\partial}{\partial t} M_{ij} \right) v^i v^j + \frac{\varepsilon}{2} \quad (6.11)$$

$$= \Delta M_{ij} v^i v^j + u^k \nabla_k M_{ij} v^i v^j + N_{ij} v^i v^j + \frac{\varepsilon}{2}. \quad (6.12)$$

The first term in the right-hand side of the equation above is non-negative, while the second term vanishes. Hence, we conclude that

$$-C\varepsilon\delta \geq N_{ij} v^i v^j < -\frac{\varepsilon}{2},$$

which produces a contradiction if δ is chosen to be sufficiently small. This proves that $M_{ij} \geq 0$ on $0 \leq t \leq \delta$. To complete the proof, just split the whole interval in pieces of length smaller than δ , and apply the result for each interval, from left to right. \square

Now, the result about preserving non-negativity of the Ricci curvature is a straightforward corollary.

Corollary 6.2.1. *Under the Ricci flow on a 3-dimensional manifold, if the initial metric g_0 has non-negative Ricci curvature, then*

$$\text{Ric}(t) \geq 0$$

for as long as the solution exists.

Proof. We apply the maximum principle for tensors, with $M_{ij} = R_{ij}$, $u^k = 0$ and $N_{ij} = -Q_{ij}$, where Q_{ij} is the tensor defined in the evolution equation for R_{ij} in (6.5). \square

We remark that we can use this form of the maximum principle for tensors because Q_{ij} is indeed a polynomial on R_{ij} , and this phenomenon does not happen in dimensions higher than 3.

A straightforward computation yields the following result:

Lemma 6.2.1. *If $R \neq 0$,*

$$\frac{\partial}{\partial t} \left(\frac{R_{ij}}{R} \right) = \Delta \left(\frac{R_{ij}}{R} \right) + \frac{2}{R} g^{pq} \nabla_p R \nabla_q \left(\frac{R_{ij}}{R} \right) - \frac{R Q_{ij} + 2S R_{ij}}{R^2},$$

where S (as in [Ham82]) is given by

$$S = g^{il} R_{ij} g^{jk} R_{kl}.$$

Now, we are in position of proving a nice upper bound for Ric:

Theorem 6.2.2. *If $R \geq 0$ and $R_{ij} \geq \varepsilon R g_{ij}$ for some $\varepsilon > 0$ at time zero, then $R_{ij}(t) \geq \varepsilon R g_{ij}(t)$ for all t such that a solution to Ricci flow exists.*

Proof. We shall make use of the maximum principle for tensors once more. We have already seen that $R > 0$ is preserved along the flow. Apply Theorem 6.2.1 for

$$\begin{aligned} M_{ij} &= \frac{R_{ij}}{R} - \varepsilon g^{ij} \\ u^k &= \frac{2}{R} g^{kl} \nabla_l R \\ N_{ij} &= 2\varepsilon R_{ij} - \left(\frac{R Q_{ij} + 2S R_{ij}}{R^2} \right). \end{aligned}$$

The previous lemma ensures that this M_{ij} evolves like in Theorem 6.2.1, but we still need to see what happens to N_{ij} when M_{ij} acquires a null-eigenvector.

At this point, it is convenient to recall we can always diagonalize the Ricci curvature at a point, and thanks to the **Uhlenbeck trick**², we can find a moving frame along the 3-manifold that preserves the orthonormal frame. The reader should be warned once more that this is only possible in dimension 3, when all compact manifolds are parallelizable. Say that the Ricci tensor can be written as

$$\text{Ric} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

²For more details, see [CK04].

such that $\lambda(0) \geq \mu(0) \geq \nu(0)$ for the initial metric.

If Ric is written in diagonal form, so are M_{ij} and N_{ij} . Without loss of generality, suppose that the null eigenvector of M_{ij} is the first eigenvector, corresponding to the eigenvalue λ of Ric . Since $R = \lambda + \mu + \nu$, we obtain the equation $\lambda = \varepsilon(\lambda + \mu + \nu)$. The corresponding entry in $R^2 N_{ij}$ is given by

$$2\varepsilon(\lambda + \mu + \nu)^2 - (\lambda + \mu + \nu)(2\lambda^2 - \mu^2 - \nu^2 - \lambda\mu - \lambda\nu + 2\mu\nu) - 2\lambda(\lambda^2 + \mu^2 + \nu^2).$$

Using the relation $\lambda = \varepsilon(\lambda + \mu + \nu)$ to get rid of ε , the equation above can be simplified to

$$(\lambda + \mu + \nu)[\lambda(\mu + \nu) + (\mu - \nu)^2] - 2\lambda(\lambda^2 + \mu^2 + \nu^2).$$

Let's put together what we know. By taking traces, our hypothesis tells us that $R \geq 3\varepsilon R$, and since $R > 0$, $\varepsilon \leq 1/3$. In this case, $(\mu + \nu) \geq 2\lambda$, showing that our choice of N_{ij} satisfies the null-eigenvector condition.

The result then follows from Theorem 6.2.1. \square

6.3 Pinching the Eigenvalues

In this section, we will see that after a while, the eigenvalues of the Ricci tensor approach each other. Intuitively, the manifold starts to become more and more round.

Consider the expression

$$S - \frac{1}{3}R^2 = (\lambda - \mu)^2 + (\mu - \nu)^2 + (\lambda - \nu)^2,$$

which measures how the eigenvalues are far away from each other.

If indeed the 3-manifold is becoming spherical, one would expect that $S - \frac{1}{3}R^2$ shrinks along the flow.

Theorem 6.3.1. *Let (M, g_0) be a 3-dimensional Riemannian manifold, such that g_0 is a metric with $\text{Ric} \geq 0$. Let $g(t)$ be a solution to the (unnormalized) Ricci flow equation (1.2) on $0 \leq t \leq T$, with initial condition $g(0) = g_0$.*

Then, there exist constants $\delta > 0$ and C , that only depend on g_0 , such that

$$S - \frac{1}{3}R^2 \leq R^{2-\delta}$$

on $0 \leq t \leq T$.

Proof. Let $\gamma = 2 - \delta$, and recall the equations for the evolution of Ric and R :

$$\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - Q_{ij} \quad (6.13)$$

$$\frac{\partial}{\partial t} R = \Delta R + 2S \quad (6.14)$$

We define $T = g^{in}g^{jk}g^{lm}R_{ij}R_{kl}R_{mn}$, and writing S and T in terms of the eigenvalues of Ric, we have

$$S = \lambda^2 + \mu^2 + \nu^2 \quad \text{and} \quad T = \lambda^3 + \mu^3 + \nu^3.$$

Finally, let

$$C = \frac{1}{2}g^{ik}g^{jk}Q_{ij}R_{kl} = \frac{1}{2}[R^3 - 5RS + 6T].$$

A simple computation gives the following result:

Lemma 6.3.1. *The expression S evolves by*

$$\frac{\partial}{\partial t} S = \Delta S - 2|\Delta_i R_{jk}|^2 + 4(T - C).$$

Using this lemma, we can state the evolution for the two terms $\frac{S}{R^\gamma}$ and $R^{2-\gamma}$:

Lemma 6.3.2. *If $R > 0$,*

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{S}{R^\gamma} \right) &= \Delta \left(\frac{S}{R^\gamma} \right) + \frac{2(\gamma-1)}{R} g^{pq} \nabla_p R \nabla_q \left(\frac{S}{R^\gamma} \right) \\ &\quad - \frac{2}{R^{\gamma+2}} |R \nabla_i R_{jk} - \nabla_i R R_{jk}|^2 \\ &\quad - \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}} S |\nabla_i R|^2 + \frac{4R(T-C) - 2\gamma S^2}{R^{\gamma+1}} \end{aligned}$$

Lemma 6.3.3. *If $R > 0$, then for any γ ,*

$$\begin{aligned} \frac{\partial}{\partial t} R^{2-\gamma} &= \Delta(R^{2-\gamma}) + \frac{2(\gamma-1)}{R} g^{pq} \nabla_p R \nabla_q (R^{2-\gamma}) \\ &\quad - \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}} R^2 |\nabla_i R|^2 + 2(2-\gamma) R^{1-\gamma} S. \end{aligned}$$

The proof of Lemmas 6.3.2 and 6.3.3 amounts to computing directly the terms involved and it is left to the reader.

Now, we want to use the maximum principle for the function

$$f = \frac{S}{R^\gamma} - \frac{1}{3} R^{2-\gamma}.$$

It is a direct consequence of the two lemmas above to see that the evolution equation satisfied by f is given by

$$\begin{aligned} \frac{\partial}{\partial t} f &= \Delta f + \frac{2(\gamma-1)}{R} g^{pq} \nabla_p R \nabla_q f - \frac{2}{R^{\gamma+2}} |R \nabla_i R_{jk} - \nabla_i R R_{jk}|^2 \\ &\quad - \frac{(2-\gamma)(\gamma-1)}{R^{\gamma+2}} \left(S - \frac{1}{3} R^2 \right) |\nabla_i R|^2 \\ &\quad - \frac{2}{R^{\gamma+1}} \left[(2-\gamma S) \left(S - \frac{1}{3} R^2 \right) - 2P \right], \end{aligned}$$

where $P = S^2 + R(C - T)$.

Clearly, P is a symmetric polynomial of degree 4 in the eigenvalues λ , μ and ν . But in fact, we can say more:

Lemma 6.3.4. *In terms of the eigenvalues of Ric , P is given by*

$$P = \lambda^2(\lambda - \mu)(\lambda - \nu) + \mu^2(\mu - \lambda)(\mu - \nu) + \nu^2(\nu - \lambda)(\nu - \mu).$$

Now, we can state the following result.

Lemma 6.3.5. *If $R > 0$ and $R_{ij} \geq \varepsilon R g_{ij}$, then*

$$P \geq \varepsilon^2 S \left(S - \frac{1}{3} R^2 \right).$$

To see this, first we simplify a bit: since both sides are homogeneous of degree 4, it suffices to check the statement on $S = (\lambda^2 + \mu^2 + \nu^2) = 1$.

Assume $\lambda \geq \mu \geq \nu > 0$. Since $(\lambda + \mu + \nu)^2 \geq 1$, $\nu \geq \varepsilon(\lambda + \mu + \nu) \geq \varepsilon$, because of the second condition in the hypothesis of the lemma. Furthermore, notice that

$$P \geq \lambda^2(\lambda - \mu)^2 + \nu^2(\mu - \nu)^2,$$

which implies

$$P \geq \varepsilon^2[(\lambda - \mu)^2 + (\mu - \nu)^2].$$

On the other hand, since

$$(\lambda - \nu)^2 \leq 2[(\lambda - \mu)^2 + (\mu - \nu)^2],$$

we see that

$$\left(S - \frac{1}{3}R^2\right) = [(\lambda - \mu)^2 + (\lambda - \mu)^2 + (\mu - \nu)^2] \leq [(\lambda - \mu)^2 + (\mu - \nu)^2] \leq P,$$

which completes the proof of the lemma.

Finally, we can state the proposition which completes, via the maximum principle, the proof of the theorem.

Proposition 6.3.1. *If $\delta \leq 2\varepsilon^2$, then*

$$\frac{\partial}{\partial t}f \leq \Delta f + u^k \nabla_k f,$$

for $u^k = \frac{2(\gamma-1)}{R}g^{kl}\nabla_l R$.

This proposition follows directly from our previous estimates.

Now we can finish the proof of Theorem 6.3.1: choose a constant C such that $f \leq C$ at time zero. Then, the maximum principle says that this bound is preserved for as long as f exists. In other words,

$$\left(S - \frac{1}{3}R^2\right) \leq CR^\gamma,$$

as claimed. \square

This theorem tells us that as long as Ric has a nice lower bound for time zero, then the Ricci flow will make the manifold rounder, by collapsing the eigenvalues together.

For restrictions on time, we will not proceed with the higher-order estimates for the Ricci tensor Ric. For a very good exposition, we refer to [CK04], or even the original paper by Hamilton [Ham82].

Chapter 7

Introduction to Kähler Geometry

In order to study the Kähler version of the Ricci Flow, that evolves metrics to the unique Kähler -Einstein metric in each Kähler class (whose interest goes far beyond Mathematics), we need to review some basic definitions and facts about complex manifolds.

7.1 Kähler manifolds

In this section, we define the basic objects we will be dealing with, while studying the Kähler Ricci flow .

Definition 7.1.1. *Let M be an n -dimensional manifold. We say that M is a **complex manifold** if it admits a system of **holomorphic coordinate charts**, that is, charts such that the transition functions are biholomorphisms.*

At each point p in a complex manifold M , we can define a map

$$J : T_p M \rightarrow T_p M, \quad J = d(z^{-1} \circ \sqrt{-1} \circ z),$$

where z is a holomorphic coordinate defined on a neighborhood of p .

It is simple to check that $J^2 = -Id$, and we call J an **almost-complex structure**. In fact, the definition of an almost-complex

structure is more general: it is simply a map $J \in \text{End}(TM)$ such that $J^2 = -Id$. We see that a complex structure induced an almost-complex structure, but the converse is not true. We say that the almost-complex structure J is **integrable** if there exists an underlying complex structure which generates it.

The **Newlander-Nirenberg Theorem** states that an almost-complex structure is integrable if the **Nijenhuis tensor**

$$N(X, Y) = [JX, JY] - J[JX, JY] - J[X, JY] - [X, Y]$$

vanishes identically.

Let M be a compact, complex manifold of complex dimension n , and consider g , a hermitian metric defined on M . Note that g is a complex-valued sesquilinear form acting on $TM \times TM$, and can therefore be written as

$$g = S - 2\sqrt{-1}\omega_g,$$

where S and $-\omega$ are real bilinear forms.

If (z_1, \dots, z_n) are local coordinates around a point $p \in M$, we can write the metric g as $\sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$. Then, it is easy to see that in these coordinates

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

The form ω_g is a real 2-form of type $(1, 1)$, and is called the *fundamental form* of the metric g .

Definition 7.1.2. *We say that a hermitian metric on a complex manifold is **Kähler** if its associated fundamental form ω_g is closed, i.e., $d\omega_g = 0$. A complex manifold equipped with a Kähler metric is called a **Kähler manifold**.*

Another characterization of a Kähler manifold M is a manifold equipped with an almost-complex structure J and a metric g such that g is J -invariant (i.e., $g(JX, JY) = g(X, Y)$) and J is parallel with respect to the Levi-Civita connection of g .

The reader can check that the conditions on J and g of a Kähler manifold implies that the Nijenhuis tensor vanishes, so any Kähler manifold is necessarily complex.

On a Kähler manifold M , the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes \mathbb{C}$ has a natural splitting. If p is a point in M , the extension of map J to $T_p M_{\mathbb{C}}$ (as a complex-linear map) has $\sqrt{-1}$ and $-\sqrt{-1}$ as eigenvalues, and we define the associated eigenspaces as $T_p^{1,0}M$ and $T_p^{0,1}M$. Hence, we have a decomposition

$$TM_{\mathbb{C}} = T^{1,0}M \otimes T^{0,1}M.$$

Definition 7.1.3. A differential $(p+q)$ -form ω is of type (p, q) if it is a section of

$$\Lambda^{p,q}M = (\Lambda^p T^{1,0}) \wedge (\Lambda^q T^{0,1}).$$

Let $z^i = x^i + \sqrt{-1}y^i$, $i = 1, \dots, n$ be complex coordinates. Then, we define

$$dz^i = dx^i + \sqrt{-1}dy^i \quad \text{and} \quad d\bar{z}^j = dx^j - \sqrt{-1}dy^j,$$

and $\frac{\partial}{\partial z^i}$, $\frac{\partial}{\partial \bar{z}^j}$, the dual of dz^i and $d\bar{z}^j$. From this definition, it is easy to see that

$$\begin{aligned} T^{1,0}M &= \text{span} \left\{ \frac{\partial}{\partial z^i} \right\}_{i=1}^n \\ T^{0,1}M &= \text{span} \left\{ \frac{\partial}{\partial \bar{z}^i} \right\}_{i=1}^n \end{aligned}$$

The exterior differentiation

$$d : \Lambda^{p,q}M \rightarrow \Lambda^{p+1,q}M \oplus \Lambda^{p,q+1}M$$

also splits according to this decomposition:

$$\begin{aligned} \partial : \Lambda^{p,q}M &\rightarrow \Lambda^{p+1,q}M \\ \bar{\partial} : \Lambda^{p,q}M &\rightarrow \Lambda^{p,q+1}M. \end{aligned}$$

Finally, if M is Kähler, we recall that the Kähler (closed) form ω_g is given, in local coordinates, by

$$\frac{\sqrt{-1}}{2} \sum_{i,j=1}^n g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

¹In fact, this whole paragraph holds for an almost-complex manifold.

where $g_{i\bar{j}} = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$.²

A last remark we make is that, due to J -invariance, the coefficients of type g_{ij} and $g_{\bar{j}\bar{l}}$ vanish identically. This cancelation phenomenon also happens for some of the coefficients of the Riemann curvature tensor. This is the subject of the next section.

7.2 Curvature and its contractions on a Kähler manifold

Many of the results in this section will be presented without a proof, mainly because they involve direct calculations using the definition.

The **Complex Christoffel symbols** are defined in analogy with the Riemannian version. We denote by $\nabla_{\mathbb{C}}$ (for simplicity, just ∇) the Levi-Civita connection of the hermitian metric g .

$$\begin{aligned} (\nabla) \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} &= \left(\Gamma_{ij}^l \frac{\partial}{\partial z^l} + \Gamma_{ij}^{\bar{l}} \frac{\partial}{\partial \bar{z}^l} \right) \\ (\nabla) \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} &= \left(\Gamma_{i\bar{j}}^l \frac{\partial}{\partial z^l} + \Gamma_{i\bar{j}}^{\bar{l}} \frac{\partial}{\partial \bar{z}^l} \right) \end{aligned}$$

A more useful expression for the Christoffel symbols is given by the following lemma.

Lemma 7.2.1. *In holomorphic coordinates, the Christoffel symbols are*

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\bar{l}} \left(\frac{\partial}{\partial z^i} g_{j\bar{l}} + \frac{\partial}{\partial \bar{z}^j} g_{i\bar{l}} - \frac{\partial}{\partial \bar{z}^j} \frac{\partial}{\partial z^l} g_{i\bar{j}} \right) = g^{k\bar{l}} \frac{\partial}{\partial z^i} g_{j\bar{l}}.$$

and $\Gamma_{ij}^k = \Gamma_{ji}^k$

All the coefficients are zero, except for the ones of the form Γ_{ij}^k or $\Gamma_{i\bar{j}}^{\bar{k}}$.

Let $R(g) = R_{i\bar{j}k\bar{l}}$ be the coordinates of the $(4,0)$ -Riemann curvature tensor of the metric g written in holomorphic coordinates. It is useful to know the following expression, writing $R(g)$ in terms of the metric g .

²Here, we are abusing notation and writing g for both the Riemannian metric and its complex extension to $TM_{\mathbb{C}}$

Lemma 7.2.2. *The components of the Kähler Riemann tensor are given by*

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} g_{k\bar{l}} + g^{u\bar{v}} \frac{\partial}{\partial z^i} g_{k\bar{v}} \frac{\partial}{\partial \bar{z}^j} g_{u\bar{l}}.$$

Note that the only non-vanishing terms have two barred indices exactly. The vanishing of the others has to do with the fact that $R(X, Y)$ is J -invariant on a Kähler manifold.

We define the **Ricci curvature tensor** of the metric g as being the trace of the Riemann curvature tensor. Its components in local coordinates can be written as

$$\text{Ric}_{k\bar{l}} = \sum_{i,j=1}^n g^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \det(g_{i\bar{j}}). \quad (7.1)$$

The **Ricci form** associated to g can then be defined by setting

$$\text{Ric} = \sum_{i,j=1}^n \text{Ric}_{i\bar{j}} dz^i \wedge d\bar{z}^j. \quad (7.2)$$

in local coordinates.

Finally, we define the **Laplacian** acting on functions to be given by

$$\Delta = g^{i\bar{j}} \nabla_i \nabla_{\bar{j}} = g^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}.$$

Further expressions concerning commutators of ∇_i , $\nabla_{\bar{j}}$ and Δ will be introduced as needed in the following chapter.

7.3 Ricci-flat metrics: the Calabi-Yau theorem

In order to motivate the search for special metrics in each Kähler class, we shall start from its original motivation.

Recall that the coordinates of the Ricci tensor are given by (7.2). The **Ricci form** associated to g can then be defined by setting

$$\text{Ric} = \sum_{i,j=1}^n \text{Ric}_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

in local coordinates. In fact, a computation shows that the Ricci form is given by

$$\text{Ric} = -\frac{\sqrt{-1}}{2\pi} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} (\log \det(g)).$$

Now, given a metric g , we can define a matrix-valued 2-form Ω by writing its expression in local coordinates, as follows

$$\Omega_i^j = \sum_{p=1}^n g^{j\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \quad (7.3)$$

This expression for Ω gives a well-defined matrix of $(1,1)$ -forms, to be called the **curvature form** of the metric g .

Following Chern-Weil Theory, we want to look at the following expression

$$\det \left(\text{Id} + \frac{t\sqrt{-1}}{2\pi} \Omega \right) = 1 + t\phi_1(g) + t^2\phi_2(g) + \dots,$$

where each $\phi_i(g)$ denotes the i -th homogeneous component of the left-hand side, considered as a polynomial in the variable t .

Each of the forms $\phi_i(g)$ is a (i,i) -form, and is called the **i -th Chern form** of the metric g . It is a fact (see for example [Wel80] for further explanations) that the cohomology class represented by each $\phi_i(g)$ is independent on the metric g , and hence it is a topological invariant of the manifold M . These cohomology classes are called the **Chern classes** of M and they are going to be denoted by $c_i(M)$.

Remark: We can define more generally the curvature $\Omega(E)$ of a hermitian metric h on a general complex vector bundle E on a complex manifold M .

Let $\nabla = \nabla(h)$ be a connection on a vector bundle $E \rightarrow M$. Then the **curvature** $\Omega_E(\nabla)$ is defined to be the element $\Omega \in \Omega^2(M, \text{End}(E, E))$ such that the \mathbb{C} -linear mapping

$$\Omega : \Gamma(M, E) \rightarrow \Omega^2(M, E)$$

has the following representation with respect to a frame f :

$$\Omega(f) = \Omega(\nabla, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

Here, $\Gamma(M, E)$ is the set of sections of the vector bundle E , $\Omega^2(M, E)$ is the set of E -valued 2-forms, and $\theta(f)$ is the connection matrix associated with ∇ and f (with respect to f , we can write $\nabla = d + \theta(f)$).

Simiarly one defines the Chern class $c_i(M, E)$ of a vector bundle and these are also independent on the choice of the connection. In fact, we use the expression “Chern classes $c_i(M)$ of the manifold M ” meaning the Chern classes $c_i(M, TM)$ of the tangent bundle of M .

We will restrict our attention to the first Chern class $c_1(M)$ of the manifold M . Note that the form $\phi_1(g)$ represents the class $c_1(M)$ (by definition), and that $\phi_1(M)$ is simply the trace of the curvature form:

$$\phi_1(g) = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^n \Omega_i^i = \frac{\sqrt{-1}}{2\pi} \sum_{i,p=1}^n g^{i\bar{p}} R_{i\bar{p}k\bar{l}} dz^k \wedge d\bar{z}^l. \quad (7.4)$$

On the other hand, notice that the right-hand side of (7.4) is equal to $\frac{\sqrt{-1}}{2\pi} \text{Ric}_{k\bar{l}}$, in view of (7.1). Therefore, we conclude that the Ricci form of a Kähler metric represents the first Chern class of the manifold M . A natural question that arises is: given a Kähler class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ in a compact, complex manifold M , and any $(1, 1)$ -form Ω representing $c_1(M)$, is that possible to find a metric g on M such that $\text{Ric}(g) = \Omega$?

This question was addressed to by Calabi in 1960, and it was answered by Yau [Yau78] almost 20 years later.

Theorem 7.3.1. (Yau, 1978) *If the manifold M is compact and Kähler, then there exists a unique Kähler metric g on M satisfying $\text{Ric}(g) = \Omega$.*

This theorem has a large number of applications in different areas of Mathematics and Physics. Its proof is based on translating the geometric statement into a non-linear partial differential equation, as follows.

First fix a Kähler form $\omega \in [\omega]$ representing the previously given Kähler class in $H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$. In local coordinates, we can write ω as $\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

The $(1, 1)$ -form Ω is a representative for $c_1(M)$, and we have seen that $\text{Ric}(\omega)$ represents the same cohomology class as Ω . Therefore,

since $\text{Ric}(\omega)$ is also a $(1,1)$ -form, we have that, due to the famous $\partial\bar{\partial}$ -Lemma, there exists a function f on M such that

$$\text{Ric}(\omega) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f,$$

where f is uniquely determined after imposing the normalization

$$\int_M (e^f - 1) \omega^n = 0. \quad (7.5)$$

Notice that f is fixed once we have fixed ω and Ω .

Again by the $\partial\bar{\partial}$ -Lemma, any other $(1,1)$ -form in the same cohomology class $[\omega]$ will be written as $\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi$, for some function $\phi \in C^\infty(M, \mathbb{R})$.

Therefore, our goal is to find a representative $\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi$ of the class $[\omega]$ that satisfies

$$\text{Ric} \left(\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right) = \Omega = \text{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f. \quad (7.6)$$

Rewriting (7.6) in local coordinates, we have

$$-\partial\bar{\partial} \log \det \left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right) = -\partial\bar{\partial} \log \det (g_{i\bar{j}}) - \partial\bar{\partial}f,$$

or

$$\partial\bar{\partial} \log \frac{\det \left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right)}{\det (g_{i\bar{j}})} = \partial\bar{\partial}f. \quad (7.7)$$

Notice that, despite of the fact that this is an expression given in local coordinates, the term at the right-hand side of (7.7) is defined globally. Therefore, we obtain an equation well-defined on all of M . In turn, this equation gives rise to the following (global) equation

$$\left(\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right)^n = e^f \omega^n. \quad (7.8)$$

We shall also require positivity of the resulting Kähler form:

$$(\omega + \partial\bar{\partial}\phi) > 0 \quad \text{on } M.$$

This equation is a non-linear partial differential equation of Monge-Ampère type, that is going to be referred to from now on as the Complex Monge-Ampère Equation.

We remark that, if ϕ is a solution to (7.8), $\omega + \partial\bar{\partial}\phi$ is the Kähler form of our target metric g , *ie*, $\text{Ric}(g) = \Omega$. Therefore, in order to find metrics that are solutions to Calabi's problem, it suffices to determine a solution ϕ to (7.8).

The celebrated Yau's Theorem in [Yau78] determines a unique solution to (7.8) when f satisfies the integrability condition (7.5). The proof of this result is based on the continuity method, and we sketch here a brief outline of the proof.

The uniqueness part of Calabi Conjecture was proved by Calabi in the 50's. Let $\omega', \omega'' \in [\omega]$ be representatives of the Kähler class $[\omega]$ such that $\text{Ric}(\omega') = \text{Ric}(\omega'') = \Omega$. Without loss of generality, we may assume that $\omega'' = \omega$, and hence $\omega' = \omega + \partial\bar{\partial}u$.

Notice that

$$0 = \frac{1}{\text{Vol}_\omega(M)} \int_M u((\omega')^n - \omega^n) \quad (7.9)$$

$$= \frac{1}{\text{Vol}_\omega(M)} \int_M -u\partial\bar{\partial}u \wedge [(\omega')^{n-1} + \quad (7.10)$$

$$(\omega')^{n-2} \wedge \omega + \cdots + \omega^{n-1}]. \quad (7.11)$$

However ω' is a Kähler form, so that $\omega' > 0$. We then conclude that the right-hand side of (7.9) is bounded from below by

$$\frac{1}{\text{Vol}_\omega(M)} \int_M -u\partial\bar{\partial}u \wedge \omega^{n-1}.$$

Therefore,

$$0 \geq \frac{1}{\text{Vol}_\omega(M)} \int_M -u\partial\bar{\partial}u \wedge \omega^{n-1} \quad (7.12)$$

$$= \frac{1}{n\text{Vol}_\omega(M)} \int_M |\partial u|^2 \omega^n \quad (7.13)$$

$$= \frac{1}{2n\text{Vol}_\omega(M)} \int_M |\nabla u|^2 \omega^n, \quad (7.14)$$

implying that $|\nabla u| = 0$, hence u is constant, proving the uniqueness of solution to (7.8).

Let us now consider the existence of solution to (7.8). Define, for all $s \in [0, 1]$, $f_s = sf + cs$, where the constant c_s is defined by the requirement that f_s satisfies the integrability condition $\int_M [e^{f_s} - 1] \omega^n = 0$.

Consider the family of equations

$$(\omega + \partial\bar{\partial}u_s)^n = e^{f_s} \omega^n. \quad (7.15)$$

We already prove that the solution u_s to (7.15) is unique, if it exists.

Let $A = \{s \in [0, 1]; (7.15) \text{ is solvable for all } t \leq s\}$. Since $A \neq \emptyset$, we just need to show that A is open and closed.

Openness: Let $s \in A$, and let t be close to s . We want to show that $t \in A$. In order to do so, let $\omega_s = \omega + \partial\bar{\partial}u_s$, for u_s a solution to (7.15). We define the operator $\Psi = \Psi_s$ by

$$\Psi : X \rightarrow Y; \quad \Psi(g) = \log \left(\frac{(\omega_s + \partial\bar{\partial}g)^n}{\omega_s^n} \right),$$

where X and Y are subsets (not subspaces) of $C^{2,1/2}(M, \mathbb{R})$ and $C^{0,1/2}(M, \mathbb{R})$ satisfying some extra non-linear conditions.

The linearization of Ψ about $g = 0$ is simply the metric laplacian with respect to the metric ω_s . By the Implicit Function Theorem, the invertibility of the laplacian (a result that can be found in [GT01], for example) establishes the claim.

Closedness: The proof that A is closed is a deep result, involving complicated *a priori* estimates. A reference for this proof is Yau's paper itself [Yau78], or for a more detailed proof, the books [Tia00] and [Ast78].

Yau's Theorem provided a satisfactory answer to the problem of finding Ricci-flat metrics when the underlying manifold M is compact.

7.4 Kähler-Einstein metrics

The simplest examples of Kähler Ricci solitons are the (static) Kähler-Einstein metrics.

Definition 7.4.1. *We say that a Kähler metric is **Kähler-Einstein** if its Ricci form is a constant multiple of the Kähler metric.*

Since Ric represents the first Chern class, a topological invariant, the existence of a Kähler-Einstein metric implies that the Chern class of the manifold has a fixed sign.

If $c_1(M) < 0$, Aubin and Yau [Aub67] proved the Calabi conjecture for negative first Chern class.

Theorem 7.4.1. (Aubin, Yau) *If a compact, complex manifold M has $c_1(M) < 0$, then there exists a Kähler-Einstein metric with negative scalar curvature. This metric is unique up to scaling.*

The situation is far more complicated if $c_1(M) > 0$. In fact, there are obstructions to the existence of a Kähler-Einstein metric in this scenario, namely the so-called **Futaki Invariant**.

Fix a metric ω such that $[\omega] = c_1(M)$. The $\partial\bar{\partial}$ -Lemma implies that there exists a smooth function f on M such that

$$\text{Ric} - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f.$$

Let $\mathcal{H}(M)$ denote the set of holomorphic vector fields on M . We define the **Futaki functional** $\mathcal{F}_{[\omega]} : \mathcal{H}(M) \rightarrow \mathbb{C}$ by

$$\mathcal{F}_{[\omega]}(V) = \int_M \langle V, \nabla f \rangle d\mu.$$

Futaki [Fut83] proved that $\mathcal{F}_{[\omega]}$ only depends on the cohomology class $[\omega]$, and its vanishing is a necessary condition for the existence of a Kähler-Einstein metric. Tian [Tia90] showed, however, that the converse is not true: he proved that some examples where $\mathcal{H}(M) = 0$, but there are no Kähler-Einstein metrics.

Chapter 8

Kähler Ricci Flow

This chapter is devoted to the study of the Ricci flow on Kähler manifolds, as first introduced by Cao [Cao85].

Last chapter, we discussed the problem of finding Kähler-Einstein metrics on a compact manifold. Recall that a necessary condition for a Kähler manifold to admit such metrics is that the first Chern Class $c_1(M)$ has a sign.

The cases $c_1(M) = 0$ and $c_1(M) \leq 0$ were settled by Yau [Yau78] and Aubin [Aub67], while we have seen that there are obstructions to the existence of a Kähler-Einstein metric if $c_1(M) \geq 0$ (the non-vanishing of the Futaki invariant).

Here, the idea of Cao [Cao85] will be discussed in detail. By flowing any Kähler metric on a compact Kähler manifold with either $c_1(M) = 0$ or $c_1(M) \leq 0$ by the Ricci flow, we obtain the (unique) Kähler-Einstein metric in the same Kähler class as the starting metric.

It should be noted here that Cao's proof relies in some results that are generalizations of Yau's higher order estimates derived in [Yau78].

8.1 Settings

Let

$$\Omega = \frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

be a fixed representative of the first Chern class $c_1(M)$, and denote, as before, the Ricci form of a Kähler metric g by

$$\text{Ric} = \frac{\sqrt{-1}}{2\pi} \text{Ric}_{i\bar{j}} dz^i \wedge d\bar{z}^j = \frac{\sqrt{-1}}{2\pi} - \partial\bar{\partial}(\log \det(g)).$$

The **Kähler Ricci flow equation** is

$$\begin{cases} \frac{\partial}{\partial t} \tilde{g}_{i\bar{j}}(t) = -\tilde{\text{Ric}}_{i\bar{j}}(t) + T_{i\bar{j}} \\ \tilde{g}_{i\bar{j}}(0) = g_{i\bar{j}} \end{cases} \quad (8.1)$$

If we can prove that the solution to (8.1) exists for all times, and converges to a limiting metric \tilde{g}_∞ , as well as show that the derivatives $\frac{\partial}{\partial t} \tilde{g}_{i\bar{j}}(t)$ converge uniformly to a constant as t approaches infinity, then \tilde{g}_∞ is the Kähler-Einstein metric we want.

8.1.1 Reduction to a scalar equation

The beauty of Kähler geometry is that sometimes, all our study relies on the Kähler potential. This is one of those happy situations.

Since both Ω and Ric lie in the same cohomology class ($c_1(M)$), the $\partial\bar{\partial}$ -Lemma tells us that there exists a smooth function f on M such that

$$\text{Ric}(\tilde{g}) - \Omega = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f.$$

We are looking for a metric $\tilde{g} = g + -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u$, for $u(t)$ defined on $M \times [0, t)$ with $u(0) = 0$, such that

$$\begin{aligned} \partial\bar{\partial} \left(\frac{\partial}{\partial t} u \right) &= -\tilde{\text{Ric}}_{i\bar{j}} + \text{Ric}_{i\bar{j}} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f \\ &= \partial\bar{\partial} \log \left(\frac{\det(g_{i\bar{j}} + -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u)}{\det(g_{i\bar{j}})} \right) + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}f \end{aligned}$$

Simplifying even more, the evolution equation for the Kähler potential u is

$$\frac{\partial}{\partial t} u = \log \left(\frac{\det(g_{i\bar{j}} + -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u)}{\det(g_{i\bar{j}})} \right) + f + c(t), \quad (8.2)$$

where $c(t)$ is a smooth function (on t) that satisfies the integrability condition

$$\int_M e^{\frac{\partial}{\partial t} u - f} dV = e^{c(t)} \text{Vol}(M).$$

Equation (8.2) is parabolic! So, short-time existence is guaranteed from standard PDE techniques, as opposed to the (real) Ricci flow equation.

In order to show long-time existence, we need to develop some *a priori* estimates of the solution up to third order. With this in hand, a slight modification of Harnack inequality will show that in fact $u(t) \rightarrow u_\infty$ uniformly, and that $\frac{\partial}{\partial t} u$ also converges uniformly to a constant.

Finally, we will discuss briefly the negative case of the Calabi Conjecture.

8.2 Long-time existence

Throughout this section, u will denote the solution to the initial value problem

$$\begin{cases} \frac{\partial}{\partial t} u = \log \det(g_{i\bar{j}} - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u) - \log(\det(g_{i\bar{j}})) + f \\ u(x, 0) = 0. \end{cases}$$

on the maximal time interval $[0, T)$, $T > 0$.

Differentiating this equation with respect to time, we obtain

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} u \right) = \tilde{g}^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left(\frac{\partial}{\partial t} u \right) = \tilde{\Delta} \left(\frac{\partial}{\partial t} u \right),$$

and applying the maximum principle, we find out that

$$\max_M \left| \frac{\partial}{\partial t} u \right| \leq \max_M |f|.$$

Lemma 8.2.1. *Let $u_{\min} = \inf_{M \times [0, T)} u$. Then, there exist constants $C_1, C_2 > 0$ such that*

$$0 < n + \Delta u \leq C_1 e^{C_2(u(t) - u_{\min})}$$

for all $t \in [0, T)$.

Proof. The first inequality comes from noticing that for all times, $\tilde{g}(t)$ is positive definite, and hence

$$\mathrm{tr}_g(\tilde{g}) > 0.$$

For the second inequality, we refer the reader to [Yau78], equation (2.24), which can be modified simply by considering the operator $\tilde{\Delta} - \frac{\partial}{\partial t}$ instead of the Laplacian. \square

8.2.1 Zeroth order estimates

Now, we proceed to the zeroth-order estimates for u . Let $v = u - \bar{u}$, where

$$\bar{u} = \frac{\int_M u dV}{\mathrm{Vol}(M)}.$$

Again, the following lemma can be derive simply from [Yau78]. For a more detailed exposition, we refer the reader to [San], which contains an analog of this result in the context of open manifolds.

Lemma 8.2.2. (Yau) *There exist positive constants C_3, C_4 such that*

$$\sup_{M \times [0, T]} v \leq C_3, \quad \sup_{M \times [0, T]} \int_M |v| dV \leq C_4.$$

Proposition 8.2.1. *There exists a constant C such that*

$$\sup_{M \times [0, T]} |v| \leq C.$$

Proof. The reader should be warned that different constants may be denoted by the same character C . Also, all the geometric objects related to \tilde{g} will be denoted with a \sim on top. Namely, $\tilde{\omega}$ will denote the Kähler form associated to the metric \tilde{g} , and so on.

The strategy of the proof is to use Nash-Moser iteration process: we want to bound L^p -norms of the function v by lower L^p -norms, inductively. Together with Lemma 8.2.2, this will imply the proposition.

The volume forms of the metrics g and \tilde{g} are given, resp., by

$$dV = \frac{\omega^n}{n!}, \quad d\tilde{V} = \frac{\tilde{\omega}^n}{n!}.$$

From the evolution equation of u , we know that

$$\frac{\partial}{\partial t} u - f = \log\left(\frac{\tilde{\omega}^n}{\omega^n}\right),$$

and hence

$$d\tilde{V} = e^{\frac{\partial}{\partial t} u - f} dV.$$

Therefore, for $p > 1$,

$$-\frac{1}{n!} \int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) = \int_M \frac{(-v)^{p-1}}{p-1} \left[e^{\left(\frac{\partial}{\partial t} u - f\right)} - 1 \right] dV.$$

On the other hand,

$$\begin{aligned} - \int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) &= \int_M (-v)^{p-2} \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} v \right) \wedge \sum_{j=1}^{n-1} \omega^j \wedge \tilde{\omega}^{n-j-1} \\ &\leq \int_M (-v)^{p-2} \left(\frac{\sqrt{-1}}{2\pi} \partial v \wedge \bar{\partial} v \right) \wedge \omega^{n-1}, \end{aligned}$$

where the last inequality follows from the fact that all the other terms involving $\omega^j \wedge \tilde{\omega}^{n-j-1}$ are non-negative.

Hence,

$$\int_M (-v)^{p-2} |\nabla v|^2 dV \leq C \int_M \leq \int_M \frac{(-v)^{p-1}}{p-1} \left[e^{\left(\frac{\partial}{\partial t} u - f\right)} - 1 \right] dV, \quad (8.3)$$

and so

$$\int_M |\nabla (-v)^{p/2}|^2 dV \leq C \frac{p^2}{p-1} \int_M (-v)^p dV. \quad (8.4)$$

So, we can estimate the H^1 -norm of $(-v)^{p/2}$ by

$$\|(-v)^{p/2}\|_{H^1}^2 = \|\nabla (-v)^{p/2}\|_{L^2}^2 + \|(-v)^{p/2}\|_{L^2}^2 \leq C \frac{p^2}{p-1} \int_M (-v)^p dV,$$

and the last term can be bounded above by $Cp^2 \int_M (-v)^p dV$ if $p > 1$.

Now, since the function v has zero average, Sobolev inequality implies that

$$\|(-v)^{p/2}\|_{L^{\frac{2n}{n-1}}}^2 \leq \|(-v)^{p/2}\|_{H^1}^2.$$

Combining the last two inequalities, we obtain, for $p > 1$,

$$\|v\|_{L^p}^p \leq Cp \|v\|_{L^p}^p. \quad (8.5)$$

The iteration happens here: let $\gamma = \frac{n}{n-1}$, and replace γ^j for p in (8.5), for $j = 0, 1, \dots$. By letting $j \rightarrow \infty$, and use the bound from Lemma 8.2.2, we obtain

$$\|v\|_{L^\infty} \leq C,$$

where the constant C is independent of time. This completes the proof of the proposition. \square

8.2.2 Higher order estimates

Combining Proposition 8.2.1 with Lemma 8.2.1, we obtain a uniform bound for the Laplacian of u with respect to g : for some constant C ,

$$n + \Delta v \leq Ce^{C(u-\bar{u})} = Ce^{C(v - \inf_{M \times [0, T]} v)} \leq C. \quad (8.6)$$

Applying Schauder estimate¹, we have the first-order estimate

$$\sup |\nabla v| \leq C (\sup |\Delta v| + \sup |v|) \leq C,$$

where the supremum in the expression above is taken over $M \times [0, T]$.

Also, note that

$$d\tilde{V} = e^{\frac{\partial}{\partial \tau} u - f} dV$$

implies that the determinant of the complex Hessian of u is uniformly bounded, and (8.6) tells us that the trace of the Hessian of u is also bounded above. This gives us the second-order bound on u , and we remark that this shows that all metrics \tilde{g} are uniformly equivalent to g .

Due to a limitation in time, we will refer the reader for the original paper of Cao [Cao85] for the proof of the third order *a priori* estimates of v . The method is a modification of the estimates in Yau's paper [Yau78].

We are now in position to prove the long-time existence of the Kähler Ricci potential u .

¹A chapter about Schauder Theory can be found in [GT01].

Theorem 8.2.1. *Let u be a solution to (8.2) on the maximum interval $[0, T)$, and let $v = u - \text{Ave}_M u$.*

Then, the C^∞ norm of v is uniformly bounded for all t , and therefore $T = \infty$.

Proof. The use of Schauder theory for the heat operator $\Delta - \frac{\partial}{\partial t}$, together with the estimates we derived in this section, will allow us to obtain all the estimates via bootstrapping.

Differentiating (8.2) with respect to z^k , we obtain

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t} \right) \left(\frac{\partial u}{\partial z^k} \right) = g^{i\bar{j}} \frac{\partial}{\partial z^k} \tilde{g}_{i\bar{j}} + \tilde{g}^{i\bar{j}} \frac{\partial}{\partial z^k} g_{i\bar{j}}. \quad (8.7)$$

The coefficients of the operator $\tilde{\Delta} - \frac{\partial}{\partial t}$ are $C^{0,\alpha}$ -bounded, as well as the right-hand side of (8.7). Therefore (see [GT01]), we see that for all k , $\frac{\partial u}{\partial z^k}$ is in $C^{2,\alpha}$, and similarly, $\frac{\partial u}{\partial \bar{z}^k}$ is also in $C^{2,\alpha}$.

But this implies that, in fact, the coefficients of the operator $\tilde{\Delta} - \frac{\partial}{\partial t}$ and the right-hand side of (8.7) are $C^{1,\alpha}$ -bounded. Schauder again implies that $\frac{\partial u}{\partial z^k}$ and $\frac{\partial u}{\partial \bar{z}^k}$ are in $C^{3,\alpha}$.

By iteration, we conclude that the C^∞ -norm of v is uniformly bounded, which shows that the Kähler Ricci potential u is defined for all times. \square

8.3 Uniform Convergence of the potential

$u(t)$

This section will be devoted to the proof of the uniform convergence of the normalized potential v , as well as to show that $\frac{\partial}{\partial t} u$ converges to a constant as $t \rightarrow \infty$.

In [Cao85], Cao used a slight generalization of the Li-Yau Harnack inequality developed in [LY86]. We state Cao's version here without a proof, since it can be derived simply from the result in [LY86].

Theorem 8.3.1. (Cao) *Let M be an n -dimensional compact mani-*

fold and let $g_{i\bar{j}}(t)$, $t \in [0, \infty)$, be a family of Kähler metrics satisfying:

- (i) $Cg_{i\bar{j}}(0) \leq g_{i\bar{j}}(t) \leq C^{-1}g_{i\bar{j}}(0)$;
- (ii) $|\frac{\partial}{\partial t}g_{i\bar{j}}|(t) \leq Cg_{i\bar{j}}(0)$;
- (iii) $R_{i\bar{j}}(t) \geq -Kg_{i\bar{j}}(0)$,

where C and K are positive constants independent of t .

Let Δ_t be the Laplacian associated to the metric $g(t)$.

If φ is a positive solution for the equation

$$(\Delta_t - \frac{\partial}{\partial t})\varphi(x, t) = 0$$

on $M \times [0, \infty)$, then for any $\alpha > 1$,

$$\begin{aligned} \sup_M \varphi(x, t_1) &\leq \inf_M \varphi(x, t_2) \left(\frac{t_1}{t_2} \right)^{\frac{n}{2}} \exp \left[\frac{1}{4(t_2 - t_1)} C^2 d^2 \right. \\ &\quad \left. + \left(\frac{n\alpha K}{2(\alpha - 1)} + C^2(n + A) \right) (t_2 - t_1) \right], \end{aligned}$$

where d is the diameter of M measured using the initial metric, $A = \sup |\nabla^2 \log \varphi|$ and $t_2 > t_1$.

Let $F = \frac{\partial}{\partial t}u$, where u is the Kähler Ricci potential. Then, F is a solution to

$$\begin{cases} (\Delta_t - \frac{\partial}{\partial t}) F = 0 \\ F(x, 0) = f(x) \end{cases} \quad (8.8)$$

By the maximum principle, we have that, for times $t_2 > t_1$,

$$\begin{aligned} \sup_M F(x, t_2) &< \sup_M F(x, t_1) < \sup_M f(x) \\ \inf_M F(x, t_2) &> \inf_M F(x, t_1) > \inf_M f(x) \end{aligned}$$

In addition, we note that the family of metrics \tilde{g} satisfies the condition in Theorem 8.3.1.

Let's define auxiliary functions

$$\varphi_n(x, t) = \sup_M F(x, n-1) - F(x, n-1+t) \quad (8.9)$$

$$\psi_n(x, t) = F(x, n-1+t) - \inf_M F(x, n-1) \quad (8.10)$$

$$\omega_n(x, t) = \sup_M F(x, t) - \inf_M F(x, t), \quad (8.11)$$

which are all positive functions and satisfy (8.8). Applying Theorem 8.3.1 to φ and ϕ (for times $t_2 = 1, t_1 = 1/2$, we obtain

$$\omega(n-1) + \omega(n - \frac{1}{2}) \leq (\omega(n-1) - \omega(n)),$$

which implies that

$$\omega(n) \leq \delta^n (\sup_M f - \inf_M f), \quad (8.12)$$

where $\delta = \frac{\gamma-1}{\gamma} < 1$.

Since the oscillation function is decreasing on t , we conclude from (8.12) that for $\delta = e^{-a}$,

$$\omega(t) \leq C e^{-at}.$$

Now, we defined the normalized derivative

$$\phi = \frac{\partial}{\partial t} u - (\text{Vol}(M))^{-1} \int_M \frac{\partial}{\partial t} u d\tilde{V}.$$

The evolution equation for ϕ is given by

$$\begin{aligned} \frac{\partial}{\partial t} \phi &= \frac{\partial^2 u}{\partial t^2} - (\text{Vol}(M))^{-1} \int_M \left[\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial t} u \tilde{\Delta} \left(\frac{\partial}{\partial t} u \right) \right] d\tilde{V} \\ &= \tilde{\Delta} \left(\frac{\partial}{\partial t} u \right) - (\text{Vol}(M))^{-1} \int_M \frac{\partial}{\partial t} u \tilde{\Delta} \left(\frac{\partial}{\partial t} u \right) d\tilde{V}, \end{aligned}$$

where the first inequality follows from recalling the evolution equation for $d\tilde{V}$:

$$\frac{\partial}{\partial t} d\tilde{V} = \frac{\partial}{\partial t} \log \det(\tilde{g}_{i\bar{j}}) d\tilde{V} = \tilde{\Delta} \left(\frac{\partial}{\partial t} u \right) d\tilde{V}.$$

Consider the quantity

$$E = \frac{1}{2} \int_M \phi^2 d\tilde{V}.$$

Some computations yield

$$\frac{\partial}{\partial t} E = \int_M (\phi - 1) |\tilde{\nabla} \frac{\partial}{\partial t} u|^2 d\tilde{V} \leq \frac{1}{2} \int_M |\tilde{\nabla} \phi|^2 d\tilde{V},$$

where last inequality follows from the fact that

$$\sup_M \phi < \omega(t) < \frac{1}{2}$$

for t sufficiently large.

Poincaré inequality applied to ϕ tells us that

$$\lambda_1(t) \int_M \phi^2 d\tilde{V} \leq \int_M |\tilde{\nabla} \phi|^2 d\tilde{V},$$

where λ_1 is the first eigenvalue of $\tilde{\Delta}(t)$. Note also that due to the uniformity of the metrics $\tilde{g}(t)$, there exists a constant c (independent of t) such that $\lambda_1(t) < c$. This implies that

$$\frac{\partial}{\partial t} E \leq -cE,$$

and hence,

$$\int_M \phi^2 dV \leq Ce^{-ct},$$

because all $d\tilde{V}$ are uniformly equivalent to dV .

Proposition 8.3.1. *The sequence of functions $v(x, t)$ (as defined in the previous section) converges uniformly, as $t \rightarrow \infty$, to a smooth function v_∞ on M . Furthermore, $\frac{\partial}{\partial t} u$ also converges to a constant.*

Proof. We will show that the family $v(x, t)$ is Cauchy in the L^1 -norm to some function $h(x)$. Recall that Theorem 8.2.1 implies that for some time sequence $t_k \rightarrow \infty$, $v(x, t_k)$ converges to a smooth function $v_\infty(x)$. So, h has no choice but to be equal to v_∞ .

Claim: $v(x, t)$ is Cauchy in the L^1 - norm.

To show this, let $0 < s < \tau$, and consider

$$\begin{aligned}
\int_M |v(x, s) - v(x, \tau)| dV &\leq \int_M \int_s^\tau \left| \frac{\partial}{\partial t} v \right| dt dV \\
&= \int_s^\tau \int_M \left| \frac{\partial}{\partial t} u - \frac{1}{\text{Vol} M} \int_M \frac{\partial}{\partial t} u \right| dt dV \\
&\leq (\text{Vol}(M))^{1/2} \int_s^\tau \left[\int_M \phi^2 dV \right]^{1/2} dt \\
&\quad + (\text{Vol}(M))^{-1} \int_s^\tau \omega(t) dt \\
&\leq C \int_s^\tau [e^{-Ct} + e^{-at}] dt.
\end{aligned}$$

At this point, we have seen that $v(x, t)$ converges in L^1 -norm to the smooth v_∞ . It is not hard to see that, in fact, the convergence happens in the C^∞ norm. Furthermore, it also follows that $\frac{\partial}{\partial t} u$ converges to a constant in the C^∞ -norm. \square

8.4 Convergence to a Kähler-Einstein metric

At this point, we proved all the ingredients to complete the proof of the main theorem of this chapter.

Theorem 8.4.1. *Let (M, g) be an n -dimensional compact Kähler manifold, and let Ω be a representative of the first Chern class $c_1(M)$ of M .*

Then, by deforming the initial metric g via the flow

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -\text{Ric}(t) + \Omega \\ g(0) = g \end{cases} \quad (8.13)$$

we obtain another Kähler metric \tilde{g} , in the same Kähler class as g , such that the Ricci curvature $\text{Ric}(\tilde{g})$ equals Ω .

A straightforward consequence is the following.

Corollary 8.4.1. *If $c_1(M) = 0$, the Kähler Ricci flow evolves any starting Kähler metric to the Ricci-flat Kähler metric in its Kähler class.*

The proof of the main theorem is immediate after proving the uniform convergence of the Kähler potential $u(t)$, and follows exactly like in Chapter 7, Section 7.3.

Finally, we remark that this method can be applied to the problem of finding Kähler-Einstein metrics on manifolds with $c_1(M) < 0$. The evolution equation for this case is given by

$$\frac{\partial}{\partial t}g(t) = -\text{Ric}(g(t)) - g(t),$$

where $g(0) = g$, and the initial metric g represents $-c_1(M)$.

The equation for the Kähler-Einstein potential is

$$\frac{\partial}{\partial t}u = \log \det(g + \partial\bar{\partial}u) - \log \det(g) + f - u.$$

The analysis for this case is simplified, as the maximum principle can be applied directly to the equation above, providing the zero-order estimate for u , and a bootstrapping argument can be used, as before, for the higher-order *a priori* estimates.

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